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**MODULATION BY RANDOM AND PSEUDO-RANDOM
SEQUENCES**

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ABSTRACT

This Report deals with the modulation of signals by discrete random and random-like sequences which may change state only at integral multiples of some basic time division t_0 . These signals may be modulated (sampled) in many fashions, depending mainly upon the types of sequences and signals available, the desired output phenomena, and the sequential rate.

In general, a sequence may sample a set of signals at random, or it may sample in some fixed deterministic fashion. Furthermore, deterministic processes may be constructed to possess certain random-like qualities. Special attention is given in this Report to random Markov chains and linear pseudo-random sequences; the signals selected for modulation are not restricted to any one class, and examples are given for sinusoids and square waves.

Specifically, the effects of carrier-signal waveform and type of sequence upon the over-all power spectrum are considered. In the case of sinusoidal modulation, the effect of phase shift is investigated.

I. BASIC CONCEPTS

The notion of having a signal or set of signals modulated by a sequence is not new in the field of communications. In fact, the whole theory of transmission of coded information by multiplexing, frequency-shift keying (FSK), CW telegraphy, etc., has used this concept for years. In CW telegraphy, a sequence (called Morse Code) chooses between a signal (carrier) and an absence of signal. In FSK systems, a sequence (called teletype) is used to choose between the two radiofrequencies which are transmitted as information. Frequency-modulated multiplex carries this process a step further, allowing a sequence (called the sampler) to gate between a number of signals so that several information channels are sandwiched together in transmission.

In this Report, a *sequence* is defined to be a discrete time function which may change states only at integral multiples of some basic time division t_0 . Furthermore, the

number a of states is finite. If these states are limited in some fashion to a set of signals which the sequence is allowed to sample, the sequence is said to *modulate* the set of sampled signals. The set of signals is called the *modulated set* of signals.

The sequences which modulate the carriers in these systems may, in general, appear random, as in the case of telegraphy, or strongly periodic, as in the case of multiplex with slowly changing channels. Sometimes, the sequences are specifically chosen to have certain properties, in order to minimize noise and other undesirable effects (Ref. 1).

An attempt has been made to treat sequence-modulated processes in general, and in such a way that the reader has a physical insight into the inner workings of the mathematical models. For this reason, a short section on

fundamentals is included. Appendices 2-3 added to the main text to provide a more rigorous mathematical basis for some of the ideas, to derive certain results, and to expand upon certain concepts which the authors consider too involved to include in the text proper.

Of basic importance in this Report is the question of the spectral distribution (Refs. 2 and 3) of such sequence-modulated devices. In particular, the spectral density of signals modulated by random Markov chains (Ref. 4) (the discrete analog of gaussian processes) and by random-like sequences which look Markovian (with respect to spectral density) are to be considered.

Section II treats the aperiodic Markov process generally, and two special cases are considered in detail. One outstanding result derived in this Section is the necessary and sufficient condition for the absence of spikes in the spectral density of a Markov process. This simple condition states that the sum of the probability-weighted sampled waveforms must vanish for all time values:

The two special cases mentioned above are processes which allow certain simplifications to be made in the spectral equation. The first of these is a process that allows the sequence to sample the signals $h_i(t)$ and $-h_i(t)$ without discrimination (that is, without bias as to which is the "positive" and which is the "negative" signal). This process, called NEP (negative equally probable), has a spectral density which is merely the weighted sum of the energy spectra of the signals over one basic time unit; there are no spectral spikes present. Such a process, while being mathematically quite simple to manipulate, is physically difficult to realize. This difficulty arises from the inability to create a sequence which changes states *instantly*; that is, a sequence which is usable in a modulation scheme must be some sort of electrical signal (see Fig. 1) and therefore has certain undesirable aspects as a result of non-zero rise times. For example, let a sinusoid carrier be multiplied by $+1$ or -1 , as dictated by a binary sequence. This process may be mechanized by a phase-modulator which swings $+90$ and -90 deg with the incoming (non-perfect) sequence voltage. The desired 180-deg change does not occur instantly; hence, the carrier goes through all intermediate phases in the transit. It is found experimentally that this does not change the over-all spectrum too greatly if the

transition is rapid; however, as the condition becomes more serious, spikes begin to appear in the spectrum.

The damaging factor in the scheme is the inability of physical equipment to mechanize a discontinuity in the slope of a signal. One obvious solution to this problem is to eliminate the discontinuity at transition times, at the same time maintaining the condition for no spectral lines. This, too, is difficult to generate, and therefore the remaining alternative is to try to *reduce* the discontinuity at transitions by requiring the signal to preserve the sign of its slope at these times. When the carrier signals are sinusoids, this means that the modulation allows a jump between frequencies only when the carrier passes through zero, and when the waveform of the new frequency also goes through zero in the same direction. This is the second type of Markov process discussed in Sec. II.

The second Markov process, called NEPS (negative equally probable-same sign slope), is one in which the over-all probability of sampling $h_i(t)$ is the same as for $-h_i(t)$; however, transitions between states are restricted in such a fashion that the slope of the modulated waveform does not change sign, so that the transition is a smoother one. Since sampling probabilities are equal, there are no lines in the NEPS spectrum, and because of the smoother changeover, the spectrum around peaks falls off more rapidly (Ref. 5) than the NEP counterpart.

Random sequences are generally easy to work with mathematically because of certain averaging procedures available. These processes allow the calculation of spectra based upon a statistical approach to a signal-modulation scheme. On the other hand, a statistically random sequence is sometimes not desirable as a modulation technique. For example, a multiplex receiver must know the code used at the transmitter to decode the incoming signal into the separate channels.

It would be desirable, then, to be able to generate some sequences *deterministically*, in such a way that some of the properties of purely random signals are inherent in them. The type of random property, of course, dictates the type of sequence which may be generated.

Pseudo-random (Ref. 6) sequences are periodic sequences which have certain random-like properties and are commonly generated by some recursive function of

past states in the sequence. When the generating logic is linear, the sequence is called a *linear recurring sequence* (Ref. 7).

The spectral density of linear sequences is found, and again two special cases are considered. These are called the NEF (negative equally frequent) and NEFS (negative equally frequent-same sign slope) processes and are merely the periodic analogs of the NEP and NEPS Markov processes. It is shown that the binary NEP and NEF spectra appear to be the same when the period of the linear sequence is long; also, the 4-level NEPS and NEFS spectra under the same restriction are greatly alike.

Examples of each process are given for modulated sine waves and square waves, and experimental curves are plotted along with theoretical calculated values.

A. Fundamentals of Signals

There is associated with each signal $y(t)$ a function $R(t, \tau)$, called its autocorrelation, and a function $G(f)$, called its power spectrum (Refs. 2 and 3). These functions are completely determined by the properties of $y(t)$ and do much in specifying the character of $y(t)$. The quantities are given by

$$R(t, \tau) = E[y(t)y(t + \tau)]$$

$$G(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} E[|Y_T(f)|^2] \quad (1)$$

where $E[X]$ is the mathematical expected value of X , and $Y_T(f)$ is the Fourier transform of the signal $y_T(t)$, defined as $y(t)$ over a range $(-T, T)$ and zero elsewhere. The first Equation indicates that $R(t, \tau)$ is a measure of the similarity between the signal at times t and $t + \tau$. It often occurs that this similarity does not depend on the starting time at all but only on the time difference τ between samples. When this is the case, $y(t)$ is said to be *stationary* (in the wide sense), and its correlation is

$$R(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T y(t)y(t + \tau) dt \quad (2)$$

This Equation is restricted, however, to signals whose time averages are the statistical expected values as stated above (the so-called ergodic property). This condition need not cause great concern, since all signals in this Report shall be assumed to have this property.

The quantity $G(f)$, on the other hand, gives a measure of the distribution of power throughout the frequency domain; that is, it indicates the frequencies at which the signal power exists. When the distribution is continuous, it is called a *continuous spectral density*, and when concentrations of power occur at discrete frequencies, $G(f)$ is said to be a *spike* (or *line*) spectrum. Often, signals possess both types of spectra.

Since both $R(\tau)$ and $G(f)$ are functions of $y(t)$, one might suspect that there exists a relation between $R(\tau)$ and $G(f)$ which excludes the signal $y(t)$ itself. This, in fact, is the case and is called the Wiener-Kintchine relation (Ref. 2):

$$G(f) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau; \quad \omega = 2\pi f$$

$$R(\tau) = \int_{-\infty}^{\infty} G(f) e^{j\omega\tau} df \quad (3)$$

This relation states that the two functions are Fourier transforms of each other. The Fourier transformation is biunique; that is, if $G(f)$ has a transform, then it has only one such transform, completely determined by $G(f)$, here designated as $R(\tau)$. Conversely, if $R(\tau)$ has a Fourier transform, it is $G(f)$. However, many signals may have the same autocorrelation function, since $R(\tau)$ is not a function of the time origin of its generating signal.

Similarly, a *cross-correlation function* may be defined in terms of two signals, $y_1(t)$ and $y_2(t)$:

$$R_{12}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T y_1(t)y_2(t + \tau) dt \quad (4)$$

The transform of this function is known as the *cross-spectral density* $G_{12}(f)$. The functions $G_{12}(f)$ and $R_{12}(\tau)$ form a Fourier-transform pair in exactly the same fashion as did $R(\tau)$ and $G(f)$ of Eq. (3). Correlation functions on a set of given signals may be put into a matrix, called the *correlation matrix*; similarly, the matrix of spectral densities is called the *spectral matrix*.

It is well known that periodic signals possess periodic correlation functions, and that the spectra of such signals are composed entirely of impulse functions at multiples of the fundamental frequency of the correlation. Furthermore, it is known that the Fourier transform of a periodic process $x(t)$ is of the form (Refs. 3 and 5)

$$X(f) = \frac{X_0(f)}{T} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right) \quad (5)$$

where T is the period of $x(t)$ and $X_0(f)$ is the Fourier transform of one cycle of $x(t)$. If $x(t)$ is assumed to be an autocorrelation function, then the corresponding periodic signal has power spectrum

$$G(f) = \frac{G_0(f)}{T} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right) \quad (6)$$

where

$$G_0(f) = \int_{-T/2}^{+T/2} R(\tau) e^{-j\omega\tau} d\tau; \quad \omega = 2\pi f \quad (7)$$

Equation (6) shows that a periodic signal has a power spectrum composed entirely of spikes, weighted by an "envelope." This envelope is the power spectrum of the aperiodic process which has identical correlation in the range $(-T/2, T/2)$, and is zero elsewhere. (If the autocorrelation were a non-zero constant outside this range, this would only change the dc power level in the spectral density, and if it were non-zero, but only approximately constant, the envelope would be in slight error, depending on the gravity of the fluctuation outside the specified range).

B. Sequence Modulation

The fundamentals described in Part A will be used in this Report basically to find specific power spectra from a device shown in Fig. 1.

Briefly, the system works as follows: At regular t_0 time intervals, a clock pulse causes a sequence generator to send the next element of the sequence to a modulator. This element of the sequence is chosen to be one of certain quantities E_1, \dots, E_a . The modulator is composed of the signals $h_1(t), \dots, h_a(t)$, which exist only in the interval $(0, t_0)$. The sequence element E_i is essentially the statement, "Let $y(t)$ in this t_0 interval be the signal $h_i(t)$, shifted to the present time." That is to say, the sequence $\{E_i\}$ may be regarded as a sequence of decisions as to which signal shall comprise $y(t)$ at any given time. In this light, the sequence may be regarded as one of Dirac delta-functions $\delta_i(t - mt_0)$, each element of which convolves the modulating signal $h_i(t)$ to the time interval $[mt_0, (m+1)t_0]$.

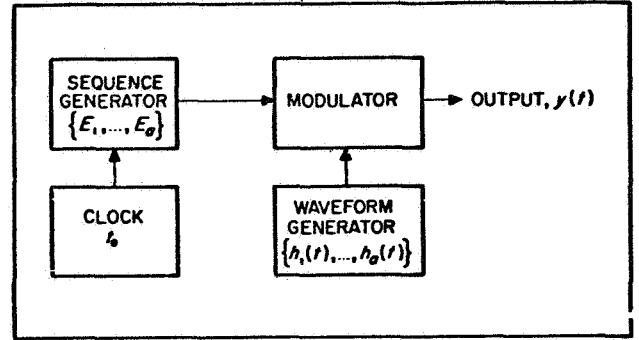


Fig. 1. Sequence Modulation Technique

For the purposes of discussion, the system, as shown in Fig. 1, may be replaced by the one in Fig. 2, in which modulated functions $\{h_i(t)\}$ are now the unit-impulse responses from the i th input to the output. The a inputs are delta functions sent from the sequence generator. The i th input signal is a series of delta functions at those times at which the sequence generator requires $h_i(t)$ to be the output signal. Designating the i th such input to be $\delta_i(t)$

$$\delta_i(t) = \sum_m \delta(t - mt_0) \quad (8)$$

where m takes on only the proper values described above.

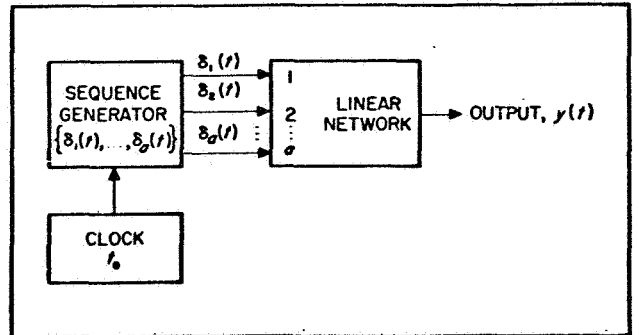


Fig. 2. Mathematical Model for Fig. 1

The transfer function between input i and the output is merely the Fourier transform of $h_i(t)$, $H_i(f)$. The network is linear, so that superposition is allowed, and the output (Ref. 3) is

$$Y(f) = \sum_{i=1}^a \Delta_i(f) H_i(f) \quad (9)$$

with $\Delta_i(f)$ the transform of $\delta_i(t)$. According to Eq. (1), the spectrum is

$$G(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \sum_{i=1}^n \sum_{k=1}^n \Delta_{i,T}^*(f) \Delta_{k,T}(f) H_i^*(f) H_k(f) \quad (10)$$

where H_i^* is the complex conjugate of H_i .

Notice the term $\lim_{T \rightarrow \infty} (1/2T) \Delta_{i,T}^*(f) \Delta_{k,T}(f)$ in this equation, which appears as some weighting function for the term $H_i^*(f) H_k(f)$. This weighting function, which shall be designated as $G_{ik}(f)$, is

$$G_{ik}(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \Delta_{i,T}^*(f) \Delta_{k,T}(f) \quad (11)$$

whose transform may be found by convolution; that is, if

$$Z(f) = X(f) Y(f)$$

then

$$z(t) = \int_{-\infty}^{+\infty} x(\tau) y(t - \tau) d\tau \quad (12)$$

The transform of $G_{ik}(f)$, therefore, is

$$R_{ik}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \delta_i(t) \delta_k(\tau + t) dt \quad (13)$$

From this it is seen that $R_{ik}(\tau)$ is a correlation function, namely, that correlation which exists between the states E_i and E_k of the sequence.

This result may also be derived in a much more rigorous fashion, as is done in Appendices A and B.

The $R_{ik}(\tau)$ are correlations between the states in a single sequence. This allows a single sequence of a states to be thought of as being a binary sequences, each zero everywhere except in that interval during which the original sequence is E_i , at which time the i th binary sequence is non-zero. Since there is really only one sequence, the $R_{ik}(\tau)$ are actually "autocorrelations" of some fashion or another, and the whole correlation matrix is also some form of correlation of the sequence with itself. However, it is difficult to assign meaning to autocorrelation of a sequence when its states are decisions. It is for this reason that it was decided to break the single sequence into a parallel binary sequences, each of which (the i th) is of the form "no E_i -yes E_i " or "0-1."

Notice in the spectral density of the output

$$G(f) = \sum_{i=1}^a \sum_{k=1}^a G_{ik}(f) H_i^*(f) H_k(f) \quad (14)$$

that all sequences which allow the products $H_i^*(f) H_k(f)$ to be weighted in the same fashion produce identical spectra. Therefore, if two sequences modulate a set of aperiodic signals $\{h_i(t)\}$ which exist only in the interval $(0, t_0)$, then the power spectral densities of the two signals are identical if the sequences have the same correlation matrix.

If the sequence is a periodic one, with all correlation functions having the same period T , then each may be written in the same form as Eq. (6), so that the power spectrum is given by the Equation

$$G(f) = \frac{1}{T} \left[\sum_{i=1}^a \sum_{k=1}^a G_{ik_0}(f) H_i^*(f) H_k(f) \right] \sum_{m=-\infty}^{+\infty} \delta\left(f - \frac{m}{T}\right) \quad (15)$$

where G_{ik_0} is the Fourier transform of one cycle of $R_{ik}(\tau)$. Notice in Eq. (15) that the term in brackets is the power spectrum of a process whose correlation functions in the range $(-T/2, T/2)$ are identical to those of the periodic process and zero outside this range (or constant, or nearly constant, depending on the degree to which a change in dc level and/or a slight deviation in envelope is acceptable).

Basically, what has been shown by this discussion is that given a time sequence, each state of which chooses a specific element of a sampled set $\{h_i(t)\}$, the power spectrum depends upon certain spectral densities of the sequence and upon the energy spectral densities of the sampled waveforms. The remainder of this Report is concerned with determining the effects of modulated set waveform and type of time sequence upon this power spectrum. There are, however, a few general remarks which can be made before these special cases are applied.

Consider a segment of a given sequence of length $2Lt_0$. During this interval, transitions between states of the sequence occur, giving rise to the numbers of transition $N_{ij}^{(n)}$, which indicate the number of times which state E_j occurs at the n th transition after E_i . Obviously, the total number of times E_i occurs in this length $2Lt_0$ is

$$N_i = \sum_{j=1}^a N_{ij}^{(n)} \quad (16)$$

In terms of these numbers, define the quantities

$$p_{ij}^{(n)} = E \left[\lim_{L \rightarrow \infty} \frac{N_{ij}^{(n)}}{N_i} \right]$$

and

$$p_i = E \left[\lim_{L \rightarrow \infty} \frac{N_i}{2L} \right] \quad (17)$$

when such quantities exist.

These quantities are the relative frequencies with which certain transitions occur and the stationary frequencies of occurrence, respectively. If the process is random, these quantities are known as the *probabilities* of transition and occurrence. If the process is periodic of length pt_0 , then the quantities $p_{ij}^{(n)}$ are also periodic, with period p at most. Specific processes are discussed in greater detail in later Sections of this Report.

The sequence correlations are

$$R_{ik}(\tau) = \lim_{L \rightarrow \infty} \frac{1}{2Lt_0} \int_{-Lt_0}^{+Lt_0} \delta_i(t) \delta_k(t + \tau) dt \quad (18)$$

These functions are impulses at those values of τ such that both τ and t are integral multiples of t_0 , E_i having occurred at time t and E_k at $t + \tau$. Obviously, if $\tau = nt_0$, the number of times this criterion is met in a given length $2L$ is $N_{ik}^{(n)}$ for τ positive, $N_{ki}^{(n)}$ for τ negative, and $N_{ii}^{(n)} = N_i$ for τ zero. Then

$$\begin{aligned} R_{ik}(\tau) &= \lim_{L \rightarrow \infty} E \left[\frac{N_i}{2Lt_0} \delta_{ik} \delta(\tau) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \frac{N_{ik}^{(n)}}{2Lt_0} \delta(\tau - nt_0) + \sum_{n=1}^{\infty} \frac{N_{ki}^{(n)}}{2Lt_0} \delta(\tau + nt_0) \right] \\ &= \frac{1}{t_0} \left\{ p_i \delta_{ik} \delta(\tau) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} p_i p_{ik}^{(n)} \delta(\tau - nt_0) + \sum_{n=1}^{\infty} p_k p_{ki}^{(n)} \delta(\tau + nt_0) \right\} \quad (19) \end{aligned}$$

The $G_{ik}(f)$ functions are

$$G_{ik}(f) = \frac{1}{t_0} \left\{ p_i \delta_{ik} + p_i \sum_{n=1}^{\infty} p_{ik}^{(n)} e^{-jn\omega t_0} + p_k \sum_{n=1}^{\infty} p_{ki}^{(n)} e^{+jn\omega t_0} \right\} \quad (20)$$

It is interesting to note from this Equation that $G_{ik}(f) = G_{ki}^*(f)$, and therefore the over-all power spectrum of the signal is

$$\begin{aligned} G(f) &= \frac{1}{t_0} \sum_{i=1}^s p_i |H_i(f)|^2 \\ &\quad + \frac{2}{t_0} \operatorname{Re} \sum_{i=1}^s \sum_{k=1}^s p_i H_i^*(f) H_k(f) P_{ik}(e^{-j\omega t_0}) \quad (21) \end{aligned}$$

where

$$P_{ik}(e^{-j\omega t_0}) = \sum_{n=1}^{\infty} p_{ik}^{(n)} e^{-jn\omega t_0} \quad (22)$$

In some cases, the time sequence generating the transitions may be partly periodic and partly aperiodic. In such cases, these parts are calculated separately.

When a substitution $z = e^{-j\omega t_0}$ is made in the functions $P_{ik}(e^{-j\omega t_0})$, the resulting functions $P_{ik}(z)$ are known as the *generating functions* of the sequence. Similarly, the matrix $P(z)$ containing these functions is the *generating matrix* of the sequence. The variable z is chosen for reasons discussed in the Part on Markov processes.

So far, the analysis given has been general and applies to any sequence which changes states at integral multiples of a basic time division t_0 . The sequence may be either periodic or aperiodic, except that use of Eq. (15) is limited to periodic sequences in which all correlation functions have equal periods T .

II. RANDOM SEQUENCE MODULATION

A. The Markov Process

Part A of the preceding Section developed the basic power spectrum of a modulated sequence without regard to the type of sequence, except that it was one of discrete states which could change at multiples of a basic time division t_0 . The discrete states themselves were not limited in any fashion, except that there were only a finite number a of them, and that each state might be construed to be some sort of decision. This part of the Report derives the power spectrum when the sequence is an aperiodic Markov chain.

The aperiodic Markov chain (Ref. 4) is a random sequence of the states E_1, \dots, E_a , such that the probabilities of sample sequences j are of the form

$$P[(E_{j_0}, E_{j_1}, \dots, E_{j_n})] = p_{j_0 j_1} p_{j_1 j_2} \dots p_{j_{n-2} j_{n-1}} p_{j_{n-1} j_n} \quad (23)$$

in terms of an initial probability distribution $\{p_k\}$ for the states $\{E_k\}$ at an arbitrary time (zero), and fixed conditional probabilities p_{ik} of E_k , given that E_i was the preceding state of the sequence. This merely indicates that at time zero there exists a certain probability distribution $\{p_k\}$ as to which of the set $\{E_k\}$ of states will be the first element of the sequence. After this, given that the sequence state at time n is E_i , the probability that E_k occurs next is p_{ik} . The over-all probability of having a certain sequence of these states occur is the product of the probability that the first occurs times the probabilities that each state is followed by its proper successor.

The transitional probabilities p_{ik} may be arranged in a matrix P , with p_{ik} in the i th row, k th column. Such a matrix is called the *transition matrix* or *stochastic matrix*. This matrix P , together with the initial distribution $\{p_k\}$, completely determines a Markov chain.

The quantity $p_{ik}^{(n)}$ is defined to be the probability that state E_k occurs at the n th transition after state E_i . Obviously, this is

$$p_{ik}^{(n)} = \sum_{j=1}^a p_{ij}^{(n-1)} p_{jk} \quad \text{or} \quad (p_{ik}^{(n)}) = P^n \quad (24)$$

The transformation $z = e^{-j\omega t_0}$ in the generating functions $P_{ik}(z)$ bears great resemblance to the z -transforms of linear sampled-data systems (Ref. 8). This concept is even more strongly supported by consideration of the Markov process on a flow-graph, as shown in Fig. 3. This Figure indicates the states E_1, \dots, E_a by a nodes and the transitional probabilities p_{ik} as arrows connecting these nodes.

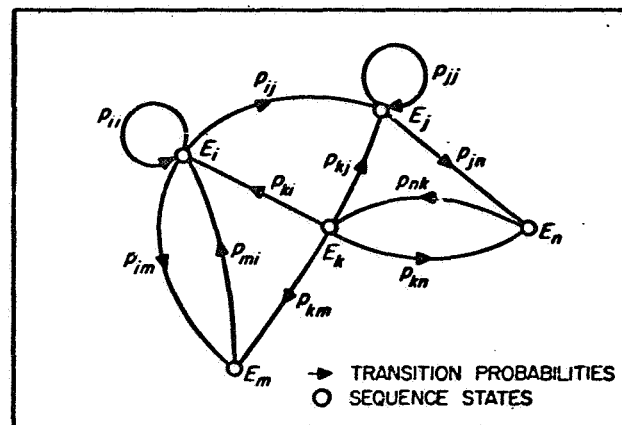


Fig. 3. Flow Graph of a Markov Process

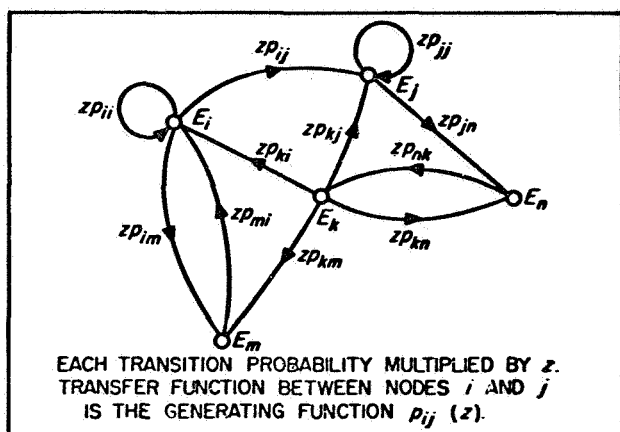
When each term of the graph is multiplied by z (Fig. 4), and a unit input to E_i is applied, the output at E_k is found to be exactly the same form as Eq. (22) (Ref. 9). This leads to the conclusion that $P_{ik}(z)$ is the transfer function between E_i and E_k when each leg of the flow-graph of the Markov process is multiplied by z .

The entire generating matrix $P(z)$ may also be found by using Eq. (24).

$$P(z) = \sum_{n=1}^{\infty} P^n z^n \quad (25)$$

By these two methods, the generating functions may be found. There are other methods which are especially useful in special problems (Ref. 4), but these are beyond the scope of this Report.

The remaining quantity to be found is the contribution of the periodic part of the correlation $R_{ik}(\tau)$ to the terms $G_{ik}(f)$. If the Markov chain is *aperiodic*, then $p_{ik}^{(n)}$ approaches p_k for large n ; that is, after many transitions,


 Fig. 4. The z -Multiplied Markov Graph

the probability that E_k will occur is not really a function of whether or not E_i ever occurred. The correlation is therefore asymptotic to

$$\lim_{|\tau| \rightarrow \infty} R_{ik}(\tau) = \frac{1}{t_0} \sum_{n=-\infty}^{+\infty} p_i p_k \delta(\tau - nt_0) \quad (26)$$

This has the spectrum of some periodic component, which, according to Eq. (6), is

$$\begin{aligned} G_{ik}(f) &= \frac{1}{t_0} \left[\frac{1}{t_0} p_i p_k \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{t_0}\right) \right] \\ &= \frac{1}{t_0^2} p_i p_k \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{t_0}\right) \end{aligned} \quad (27)$$

Combination of this result with Eqs. (25) and (21) gives the complete power spectrum of the aperiodic Markov process:

$$\begin{aligned} G(f) &= \frac{1}{t_0^2} \sum_{n=-\infty}^{+\infty} \left| \sum_{i=1}^n p_i H_i\left(\frac{n}{t_0}\right) \right|^2 \delta\left(f - \frac{n}{t_0}\right) \\ &+ \frac{1}{t_0} \sum_{i=1}^n p_i |H_i(f)|^2 \\ &+ \frac{2}{t_0} \operatorname{Re} \left[\sum_{i=1}^n \sum_{k=1}^n p_i H_i^*(f) H_k(f) P_{ik}(e^{-j\omega t_0}) \right] \end{aligned} \quad (28)$$

The notation used in Eq. (28) is:

$\delta(f)$ = the Dirac delta function
 p_1, \dots, p_n = the stationary probability distribution of the Markov chain

$p_{ij}^{(n)}$ = the probability that the signal $b_j(t)$ occurs at the n th transition after the occurrence of the signal $b_i(t)$

$$P_{ij}(z) = \sum_{n=1}^{\infty} p_{ij}^{(n)} z^n$$

$$H_i(f) = \int_0^{t_0} b_i(t) e^{-j2\pi f t} dt$$

$b_1(t), \dots, b_n(t)$ = the set of modulated signals

Notice that the first term (spike spectrum) of this equation vanishes when

$$\sum_{i=1}^n p_i H_i(f) = 0 \quad (29)$$

which implies that a necessary and sufficient condition for the absence of spectral spikes is that

$$\sum_{i=1}^n p_i b_i(t) = 0 \quad (30)$$

The discussion here has been devoted to the physical aspect of the Markov spectrum. A more rigorous treatment may be found in Appendices A and B.

The modulated set may be regarded in the frequency domain as a modulated vector $H(f)$. This vector has a conjugated, weighted transpose $[pH^*(f)]^T$. With these notations, Eq. (28) may be rewritten

$$\begin{aligned} G(f) &= \frac{1}{t_0^2} \left| \sum_{i=1}^n p_i H_i(f) \right|^2 \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{t_0}\right) + \frac{1}{t_0} [pH^*(f)]^T H(f) \\ &+ \frac{2}{t_0} \operatorname{Re} \{ [pH^*(f)]^T P(e^{-j\omega t_0}) H(f) \} \end{aligned} \quad (31)$$

Basically, there are two types of signals $y(t)$ which will be discussed in the remainder of this section. These two classes of signals are defined in terms of the manner by which they are generated.

Each class is defined in such a way that certain symmetries exist, in order that certain simplifications of the spectral equation may be made. The first class is designated "negative equally probable," and consists of that class of Markov processes for which the first and third terms of Eq. (28) vanish. The second class, designated "negative equally probable, same sign slope," consists of those Markov processes for which Eq. (30) is satisfied (no spectral spikes), with transitions arranged so that the slopes at the transition times do not change sign. The

behavior of those processes will become more apparent upon detailed study of each in turn.

B. The NEP Process

A signal $y(t)$ is said to be an NEP (negative equally probable) process if (1) for each element $h_i(t)$ of the modulating set $\{h_i(t)\}$ of a Markov process, $-h_i(t)$ is also in the set, and (2) the stationary probabilities on $h_i(t)$ and $-h_i(t)$ are equal; also, the transitional properties of $h_i(t)$ are the same as those of $-h_i(t)$. That is, $p_{jk} = p_{rs}$ whenever $h_j(t) = \pm h_r(t)$, and $h_k(t) = \pm h_s(t)$.

This process has been specifically defined so that there are no spectral spikes and so that the transitional properties themselves do not affect the over-all spectrum. In this case, as a direct result of substitution in Eq. (28), the spectrum is

$$G(f) = \frac{1}{t_0} \sum_{i=1}^a p_i |H_i(f)|^2 \quad (32)$$

Note from this equation that the over-all NEP spectrum is merely the weighted sum of the energy spectra of each individual $h_i(t)$ in the modulated set. It is sufficient, therefore, to have a knowledge of the behavior of each component $p_i |H_i(f)|^2$ in order to predict the spectral density of the whole process. This is not true, in general, for the more complicated NEPS process.

Example 1: NEP Sinusoids

Consider the NEP process, which has the following properties:

$$h_i(t) = \sin(\omega_i t + \phi_i)$$

$$p_{ik} = \frac{1}{a} = p_i$$

$$\omega_i = \frac{n_i \pi}{t_0}$$

$$\omega = 2\pi f$$

with n_i integral; then, by Fourier transformation,

$$H_i(f) = \frac{\omega_i \cos \phi_i + j\omega \sin \phi_i}{\omega_i^2 - \omega^2} 2(j)^{(n_i+1)} \left[\sin \left(\frac{\omega - \omega_i}{2} \right) t_0 \right] e^{(j\omega t_0)/2} \quad (33)$$

and, therefore,

$$G(f) = \frac{t_0}{a} \sum_{i=1}^a \left[\frac{\sin \left(\frac{\omega - \omega_i}{2} \right) t_0}{\left(\frac{\omega - \omega_i}{2} \right) t_0} \right]^2 \left[\frac{\cos^2 \phi_i + \left(\frac{\omega}{\omega_i} \right)^2 \sin^2 \phi_i}{\left(1 + \frac{\omega}{\omega_i} \right)^2} \right] \quad (34)$$

Note here that the phase angle ϕ_i , by which the sinusoid is shifted with respect to the Markov sequence, is of great importance in the region ($\omega > \omega_i$). In fact, for any ϕ_i , each term of Eq. (34) is enveloped by

$$G_{env}(f) = \frac{4}{a t_0} \frac{\omega_i^2 \cos^2 \phi_i + \omega^2 \sin^2 \phi_i}{(\omega^2 - \omega_i^2)^2} \quad (35)$$

The Equation shows that if ϕ_i is non-zero, or is not an integral multiple of π , then the spectrum ultimately decreases 6 db/octave. When ϕ_i is zero or π , the spectrum falls off at 12 db/octave. Note, however, that the rate of approach to the 6-db asymptote is determined by ϕ_i , in that for ϕ_i near zero (but non-zero), the spectrum seems to approach the 12-db limit; but as ω becomes sufficiently large, it ultimately changes over to the 6-db limit. That is to say, the density function chooses a cross-over frequency ω_c at which it changes from a 12-db to a 6-db/octave asymptote. This occurs at

$$\omega_c^2 = \omega_i^2 (1 + \csc^2 \phi_i)$$

or

$$\omega_c = \pm \omega_i \sqrt{1 + \csc^2 \phi_i} \quad (36)$$

The curve is asymptotic to 12 db/octave up to ω_c , beyond which it then becomes asymptotic to 6 db/octave. In the two limiting cases, $\phi_i = 0$ and $\phi_i = \pi/2$, the curves have only one asymptote, because ω_c is either at infinity or at $\omega_i \sqrt{2}$.

Random phase. When each ϕ_i is considered to be a random variable, uniformly distributed over the range $(0-2\pi)$, the resulting spectrum is the average over this range; that is, each term is of the form:

$$G_i(f) = \int_{-\infty}^{\infty} G_i(f, \phi_i) p(\phi_i) d\phi_i = \int_0^{2\pi} \frac{1}{2\pi} G_i(f, \phi_i) d\phi_i \\ = \frac{t_0}{2} \left[\frac{\sin \left(\frac{\omega - \omega_i}{2} \right) t_0}{\left(\frac{\omega - \omega_i}{2} \right) t_0} \right]^2 \left[\frac{1 + \left(\frac{\omega}{\omega_i} \right)^2}{\left(1 + \frac{\omega}{\omega_i} \right)^2} \right] \quad (37)$$

This result is important when compared with the results of the next part of the example.

Convolution of spectra. Let the signal $y(t)$ be generated by multiplying a sinusoid $x(t)$ by a Markov sequence $m(t)$ of ones and minus ones. This signal, $y(t) = x(t)m(t)$, has autocorrelation

$$R_y(\tau) = E[x(t)x(t+\tau)m(t)m(t+\tau)] \quad (38)$$

Let it be assumed that $x(t)$ and $m(t)$ are independent; if such is the case, then the above equation factors into

$$R_y(\tau) = E[x(t)x(t+\tau)]E[m(t)m(t+\tau)] = R_x(\tau)R_m(\tau) \quad (39)$$

It is well known that functions which multiply in the time domain convolve in the frequency domain:

$$G_y(f) = \int_{-\infty}^{\infty} G_x(\zeta)G_m(f-\zeta)d\zeta \quad (40)$$

The Markov sequence itself has spectrum

$$G_m(f) = t_0 \left(\frac{\sin \frac{\omega t_0}{2}}{\frac{\omega t_0}{2}} \right)^2 \quad (41)$$

and the sinusoid has spectrum

$$G_x(f) = \frac{1}{4} [\delta(f-f_1) + \delta(f+f_1)] \quad (42)$$

Their convolution is

$$G_y(f) = \frac{t_0}{2} \left[\frac{\sin \left(\frac{\omega - \omega_1}{2} \right) t_0}{\left(\frac{\omega - \omega_1}{2} \right) t_0} \right]^2 \left[\frac{1 + \left(\frac{\omega}{\omega_1} \right)^2}{\left(1 + \frac{\omega}{\omega_1} \right)^2} \right] \quad (43)$$

if $\omega_1 = \pi n/t_0$, with n integral. Comparison of Eq. (43) with the results of Eq. (34) shows that the two functions $x(t)$ and $m(t)$ are not really uncorrelated, but for $\omega \gg \omega_1$, and $\phi \neq 0$, the same shape spectrum is obtained.

In fact, it is seen that when ϕ_i is equal to $\pi/4$, Eqs. (34) and (43) are the same; also, Eq. (43) corresponds exactly to the case of Eq. (37), where ϕ_i is a random variable, uniformly distributed over the range $(0-2\pi)$.

Example 2: NEP Square Waves

Let $h_i(t)$ be a unit square wave of n_i half-cycles per t_0 , and $p_i = 1/a$. Then, designate $\omega_i = n_i\pi/t_0$. In the frequency domain,

$$H_i(f) = \frac{t_0}{n_i} \left(\frac{\sin \frac{\omega t_0}{2n_i}}{\frac{\omega t_0}{2n_i}} \right) e^{-j(\omega t_0 + 2n_i)} \sum_{k=0}^{n_i-1} e^{-j(\omega t_0 + n_i\pi)k/n_i} \quad (44)$$

The power spectrum is given by

$$G(f) = \frac{t_0}{a} \sum_{i=1}^a \frac{1}{n_i^2} \left[\frac{\sin \frac{\pi}{2} \left(\frac{\omega}{\omega_i} \right)}{\frac{\pi}{2} \left(\frac{\omega}{\omega_i} \right)} \right]^2 \times \left[n_i + 2 \sum_{q=1}^{n_i} (n_i - q) \cos \left(\frac{\omega - \omega_i}{\omega_i} \pi q \right) \right] \quad (45)$$

Note that this is asymptotic to 6 db/octave when $\omega = 2\pi f$ is large.

The spectra for single NEP sinusoids and square waves are plotted in Fig. 5 for comparison. Note that the 90-deg-shifted sinusoid distribution falls off at approximately the same rate as the square wave.

C. The NEPS Process

A signal $y(t)$ is said to be an NEPS (negative equally probable, same sign slope) process if (1) both $h_i(t)$ and $-h_i(t)$ are in $\{h_i(t)\}$, their stationary probabilities are equal, and the transitional properties are such that $p_{ij} = p_{rs}$ if $h_i(t) = -h_r(t)$ and $h_j(t) = -h_s(t)$; (2) $h_i(0) = h_i(t_0) = 0$; and (3) the slope of $y(t)$ does not change its sign at transitions.

By these axioms, the NEPS process not only has an absence of spectral lines but also provides the possibility of a smoother transition between waveforms in adjacent positions of the output. The process has been defined so that the slopes at $t = 0$ and $t = t_0$ exclude certain transitions. The slopes at cross-over times allow the set $\{h_i(t)\}$ to be partitioned into 4 subsets:

$$\begin{aligned} \{b_a\} &= \{b_i(t); \text{slope } (+) \text{ at } t = 0, (+) \text{ at } t_0\} \\ \{b_b\} &= \{b_j(t); \text{slope } (+) \text{ at } t = 0, (-) \text{ at } t_0\} \\ \{b_c\} &= \{-b_a\} \\ \{b_d\} &= \{-b_b\} \end{aligned} \quad (46)$$

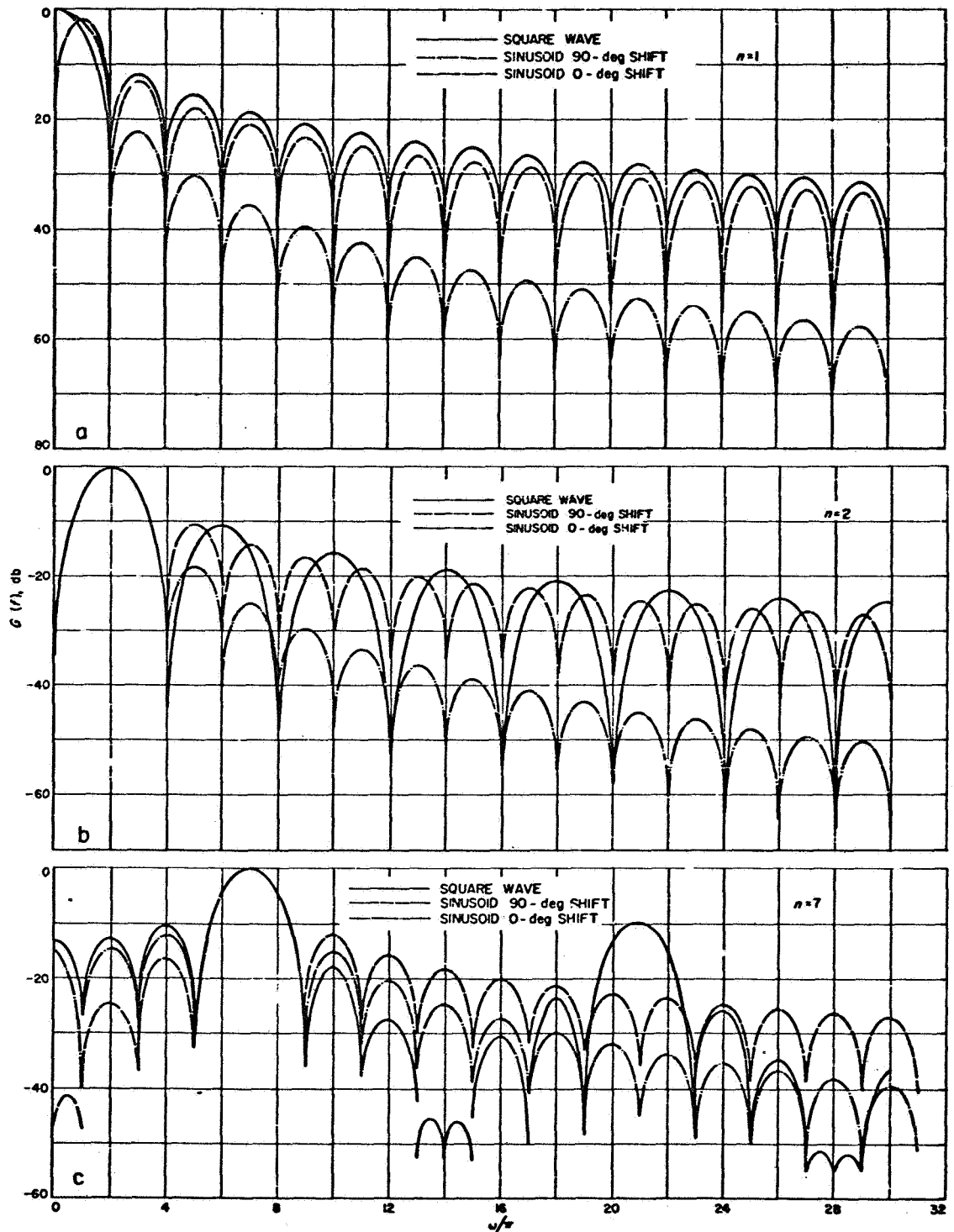


Fig. 5. Comparison of Binary NEP (Random Process) Spectral Densities for Sampled Sets of Square Waves and Sinusoids for Various Values of n_i

The transition matrix P is partitioned in the same fashion:

$$P = \begin{bmatrix} \alpha & \beta & 0 & 0 \\ 0 & 0 & \gamma & \epsilon \\ 0 & 0 & \alpha & \beta \\ \gamma & \epsilon & 0 & 0 \end{bmatrix} = \begin{bmatrix} B & C \\ C & B \end{bmatrix} \quad (47)$$

The NEPS-process power spectrum (see Eq. 31) is given by the Equation

$$G(f) = \frac{2}{t_0} \sum_{i=1}^{q/2} p_i |H_i(f)|^2 + \frac{4}{t_0} \operatorname{Re} \left\{ [p_\alpha H_\alpha^* p_\beta H_\beta^*] E(z) \begin{bmatrix} H_\alpha \\ H_\beta \end{bmatrix} \right\} \quad (48)$$

which involves positive waveforms α and β only, with

$$E = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\epsilon \end{bmatrix}, \quad z = e^{-j2\pi f t_0}, \quad \text{and} \quad E(z) = \sum_{n=1}^{\infty} E^n z^n \quad (49)$$

This result is derived in Appendix C.

As a special application, consider the case with all eligible states equally likely (all non-zero elements in P are $2/a$). The matrix E is composed of q rows of $+2/a$ and r rows of $-2/a$; that is, there are q "even" waveforms and r "odd" waveforms. Direct calculation shows that

$$E^n = (q-r)^{n-1} \left(\frac{2}{a}\right)^{n-1} E = \left(\frac{q-r}{q+r}\right)^{n-1} E \quad (50)$$

The matrix $E(z)$ is the converging series

$$E(z) = zE \sum_{n=0}^{\infty} \left(\frac{q-r}{q+r}\right)^n z^n = \frac{zE}{1 - \left(\frac{q-r}{q+r}\right)z} \quad (51)$$

Thus, the third term of Eq. (31) is

$$p H^* P(z) H = \frac{4}{a^2} \left[\frac{z}{1 - \left(\frac{q-r}{q+r}\right)z} \right] \left[\sum_{\alpha} H_{\alpha}^*(f) - \sum_{\beta} H_{\beta}^*(f) \right] \times \left[\sum_{\alpha} H_{\alpha}(f) + \sum_{\beta} H_{\beta}(f) \right] \quad (52)$$

This NEPS process, therefore, has spectral density

$$G(f) = \frac{2}{at_0} \sum_{i=1}^{q/2} |H_i(f)|^2 + \frac{8}{a^2 t_0} \times \operatorname{Re} \left\{ \left[\sum_{\alpha} H_{\alpha}^*(f) - \sum_{\beta} H_{\beta}^*(f) \right] \left[\sum_{\alpha} H_{\alpha}(f) + \sum_{\beta} H_{\beta}(f) \right] \times \left[\frac{e^{-j2\pi f t_0}}{1 - \left(\frac{q-r}{q+r}\right)e^{-j2\pi f t_0}} \right] \right\} \quad (53)$$

with sums only over positive waveforms α and β . For the simple case for which $q = r$ (equal number of waveforms in $\{h_{\alpha}(t)\}$ and $\{h_{\beta}(t)\}$), the spectrum reduces to

$$G(f) = \frac{2}{at_0} \sum_{i=1}^{q/2} |H_i(f)|^2 + \frac{8}{a^2 t_0} \times \operatorname{Re} \left\{ \left[\sum_{\alpha} H_{\alpha}^*(f) - \sum_{\beta} H_{\beta}^*(f) \right] \times \left[\sum_{\alpha} H_{\alpha}(f) + \sum_{\beta} H_{\beta}(f) \right] e^{-j2\pi f t_0} \right\} \quad (54a)$$

Clearly, the NEPS process is defined in such a way that the modulated set cannot be composed entirely of even waveforms, for if it were, the inverses in the set would never be sampled. However, they may all be odd waveforms; in this case, $q = 0$:

$$G(f) = \frac{2}{at_0} \left[\sum_{\beta} |H_{\beta}(f)|^2 - \frac{2}{a} \left| \sum_{\beta} H_{\beta}(f) \right|^2 \right] \quad (54b)$$

Example 3: NEPS Sinusoids

Consider a NEPS process described by

$$b_i(t) = \sin(\omega_i t) \\ a = 4, \text{ with} \\ \omega_i = \frac{n_i \pi}{t_0}$$

so that n_1 is even and n_2 odd. The transition matrix P is

$$P = \frac{1}{4} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad (55)$$

Evaluating Eq. (54a) for this case,

$$G(f) = \frac{1}{2t_0} \left\{ |H_1(f)|^2 (1 + \cos \omega t_0) + |H_2(f)|^2 (1 - \cos \omega t_0) - j[H_1^*(f)H_2(f) - H_2^*(f)H_1(f)] \sin \omega t_0 \right\} \quad (56)$$

The $H_i(f)$ are given by Eq. (33); the spectrum is

$$G(f) = \frac{1}{t_0} \left[\left(\frac{\omega_1}{\omega^2 - \omega_1^2} \right) - \left(\frac{\omega_2}{\omega^2 - \omega_2^2} \right) \right]^2 \sin^2 \omega t_0 \quad (57)$$

Notice here that as ω grows large, $G(f)$ decreases approximately 12 db/octave, and that $G(f)$ falls off most rapidly near the fundamental peaks if n_1 and n_2 are adjacent integers.

Example 4: NEPS Square-Waves

Consider the case described by the matrix in Eq. (55), except let the modulation be unit square waves of n_1 and n_2 half-cycles per t_0 period. The $H_i(f)$ are given in Eq. (44). Upon substitution of these values into Eq. (56), it is found that the spectrum of this NEPS square-wave process is

$$G(f) = t_0 \left\{ \sum_{i=1}^2 \frac{\cos^2 \left(\frac{\omega - \omega_i}{2} \right) t_0 \left[\frac{\sin \frac{\pi}{2} \left(\frac{\omega}{\omega_i} \right)}{\frac{\pi}{2} \left(\frac{\omega}{\omega_i} \right)} \right]^2}{n_i^2} \right. \\ \left. \times \left[n_i + 2 \sum_{q=1}^{n_i} (n_i - q) \cos \left[\left(\frac{\omega - \omega_i}{\omega_i} \right) \pi q \right] \right] \right\} \\ - \frac{t_0}{n_1 n_2} \sin \omega t_0 \left[\frac{\sin \frac{\pi}{2} \left(\frac{\omega}{\omega_1} \right)}{\frac{\pi}{2} \left(\frac{\omega}{\omega_1} \right)} \right] \left[\frac{\sin \frac{\pi}{2} \left(\frac{\omega}{\omega_2} \right)}{\frac{\pi}{2} \left(\frac{\omega}{\omega_2} \right)} \right] \\ \times \sum_{k=0}^{n_1-1} \sum_{m=0}^{n_2-1} \sin \left[\left(\frac{2m+1}{n_2} - \frac{2k+1}{n_1} \right) \frac{\omega t_0}{2} + (m-k)\pi \right] \quad (58)$$

This formidable Equation has been solved by a digital computer for several values of n_1 and n_2 (the curves are shown in Fig. 9).

D. Observation of Spectral Spikes

As may be seen from Eq. (28), the spectrum may, in general, be thought of as being composed of two parts: a spike-spectrum $G_s(f)$, and a continuous spectrum $G_c(f)$. In both the NEP and NEPS, probabilities have been chosen to eliminate $G_s(f)$. However, now consider the case for which the weighted sum of the waveforms does not vanish. In particular, if the stationary probabilities between a waveform $h_i(f)$ and its negative differ by an amount ϵ , then the continuous spectrum does not change appreciably if ϵ is small. The power contained in a given band Δf , directly attributable to $G_s(f)$, as compared to the continuous spectrum in this band, is

$$\lambda(f, \Delta f) = \frac{\int_{\Delta f} G_s(f) df}{\int_{\Delta f} G_c(f) df} \quad (59)$$

Here it is assumed that only one spike lies in the Δf range. A receiver sees the spectrum as

$$G_{rec}(f) \simeq (1 + \lambda) G_c(f) \quad (60)$$

In order to evaluate this Equation to obtain an order-of-magnitude picture of the spectrum where spikes should appear, let $\{h_i(t)\}$ contain only $\pm h_i(t)$, and assume $G_c(f)$ to be fairly constant within the Δf interval. The ratio for an NEP process is

$$\lambda \simeq \frac{\epsilon^2}{t_0 \Delta f} \quad (61)$$

if ϵ is small. This says that at points of the continuous spectrum (when $G_c(f)$ is about constant), the deviation due to spikes is small if ϵ is small.

This analysis has considered that the time of integration in a given bandwidth Δf was infinite. Frequency analyzers which publish spectra based on short integrations may, therefore, show great deviation from the expected behavior discussed here. In order to obtain reasonable looking results from an analyzer, integration time must be long compared to the reciprocal of the analyzer bandwidth.

III. PSEUDO-RANDOM MODULATION

A. Random-like Sequences

Pseudo-random sequences are sequences which possess certain qualities of randomness, yet are deterministic and periodic in nature. Specifically, their properties are such that the sequence appears to be some random sequence (such as Markov). These random sequences may be generated in many fashions, most commonly by recurrence techniques applied to shift registers. If such a sequence is binary, it is known as a PN (pseudo-noise) sequence.

In restricting the discussion here to periodic sequences, it is evident that no continuous spectrum will be present in $G(f)$; rather, power will be concentrated at discrete frequencies. Also, these periodic sequences (as yet undefined) are such that, in general, $P^n \neq (p_{ik}^{(n)}) = P^{(n)}$, as was true for the Markov chain. The quantities $p_{ik}^{(n)}$ are now relative frequencies of transitions per period.

In order that a sequence be pseudo-random, it is necessary first to specify relative frequencies of occurrence for each state in the sequence; second, one must specify the type of transitions which may be made between the states; and third, some property must be assigned to the sequence in order that it appear random to a certain degree.

Although these specifications are usually interrelated, the first, relative frequency, is usually fixed by the manner in which the sequence is generated. The second, transition, is dependent upon what modulation characteristics are desired and what randomness properties are to be assigned to the sequence. The random properties are conveniently established by the correlation functions $R_{ik}(\tau)$, as described in Sec. IA. The correlation matrix $R(nt_n)$ has heretofore been given the notation $(p_i p_{ik}^{(n)})$ for convenience.

If it is desired that the sequence appear Markovian, then it is merely necessary to insure that the $R_{ik}(\tau)$ of the two sequences have approximately¹ the same shape over a range $(-T/2 \leq \tau \leq +T/2)$, outside of which the non-periodic correlations are zero. As discussed previously, non-zero values of correlation outside this range, if con-

stant or fairly constant, do not disrupt the spectra greatly, except for causing some discrepancy in the dc levels.

The only problem which arises in characterizing a pseudo-random process (which is to appear Markovian) is the approximation of the conditions given above. Luckily, there are several methods by which this may be attempted. Perhaps most widely known are those methods which employ linear recurrence techniques to a -level shift registers. Several non-linear techniques are also known, which exhibit many desirable characteristics.

For certain Markov processes, the corresponding periodic process is easily found. For example, a binary Markov sequence which has two-level correlation is easily approximated by the PN sequence which also possesses two-level correlation. In other cases, however, the approximation is more difficult and somewhat more crude. For these reasons, the particular sequence which gives a Markov envelope to spectral lines must be chosen carefully.

B. Linear Recurring Sequences

In this Part, the power density function of a certain type of sequence is developed. In general, the sequence does not behave properly to produce Markov properties. However, in certain cases, very desirable results can be obtained by using such a sequence to modulate a set $\{h_i(t)\}$.

Given a sequence, $\{b_i\}$, and a set of coefficients, (c_0, c_1, \dots, c_m) , each composed of elements over a finite field K , the sequence is said to be *linearly recurring* if all segments of the sequence with length $m + 1$ satisfy the relation

$$\sum_{i=0}^m c_i b_{n-i} = 0 \quad (62)$$

Such a sequence is easily mechanized by shift registers, as shown in Fig. 6.

Much work (Refs. 6 and 7) has been done on such sequences, and an abundance of information is available on the subject. The major portion of the theory is beyond the scope of this report; however, a few significant properties bear discussion.

¹Recent work at the Jet Propulsion Laboratory has shown that no periodic sequence may satisfy this property exactly.

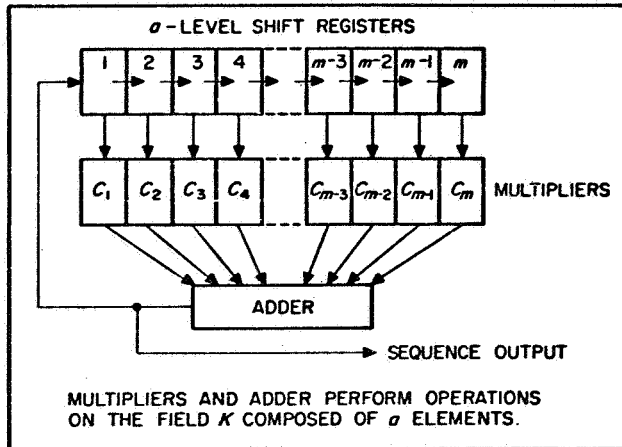


Fig. 6. Method for Generating Linear Recurring Sequences

First, due to the fact that the elements of the sequence lie in a finite field, K , the number of states $E_1 \cdots E_a$ must be a power of a prime. In particular, if the number of states is a , then K is the field with a elements, and $a = q^k$ for some prime q and positive integer k .

Such a linear sequence which is of maximum length p is called an m -sequence, of level a and period p . The m -sequence which satisfies Eq. (62) has period $p = a^m - 1$, and all states except one occur a^{m-1} times during this period. The excepted state is the one corresponding to the "zero" of the finite field, which only occurs $a^{m-1} - 1$ times; this state shall be designated E_a .

The subsequences $\{E_{i_1} \cdots E_{i_k}\}$, k elements long ($k \leq m$), each occur a^{m-k} times, except those in which all E_i are E_a . Those occur $a^{m-k} - 1$ times per period.

Another significant property is that the sequence formed by adding to each term b_i of a given m -sequence that term of the sequence which is translated by r , b_{i+r} , is the same sequence translated by some integer s ; i.e.,

$$b_{i+s} = b_i + b_{i+r} \quad (63)$$

This is commonly called the "cycle-and-add" property.

The frequencies of occurrence (designating E_a as the state corresponding to terms $b_k = 0$) are

$$\begin{aligned} p_i &= \frac{a^{m-1}}{a^m - 1} = \frac{p+1}{ap}; \quad (i = 1, 2, \dots, a-1) \\ p_a &= \frac{a^{m-1} - 1}{a^m - 1} = \frac{p+1-a}{ap} \end{aligned} \quad (64)$$

and the transition matrix is

$$P = \begin{bmatrix} \frac{1}{a} & \frac{1}{a} & \cdots & \frac{1}{a} \\ \frac{1}{a} & \frac{1}{a} & \cdots & \frac{1}{a} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a^{m-2}}{a^{m-1}-1} & \frac{a^{m-2}}{a^{m-1}-1} & \cdots & \frac{a^{m-2}-1}{a^{m-1}-1} \end{bmatrix} \quad (65)$$

The transition matrices $P^{(n)}$ are equal for all $n \not\equiv 0 \pmod{s}$, $s = p/(a-1)$. When $n \equiv 0 \pmod{s}$, the matrices $P^{(n)}$ are composed of elements

$$p_{ik}^{(n)} = \delta(k, \lambda^r i) \quad (66)$$

(see Fig. 7) for some primitive element λ of the field (both i and k are also members of this field), and $\delta(k, \lambda^r i)$ is the Kronecker delta. The primitive λ is the element

$$\lambda = b_{i+s} b_i^{-1}$$

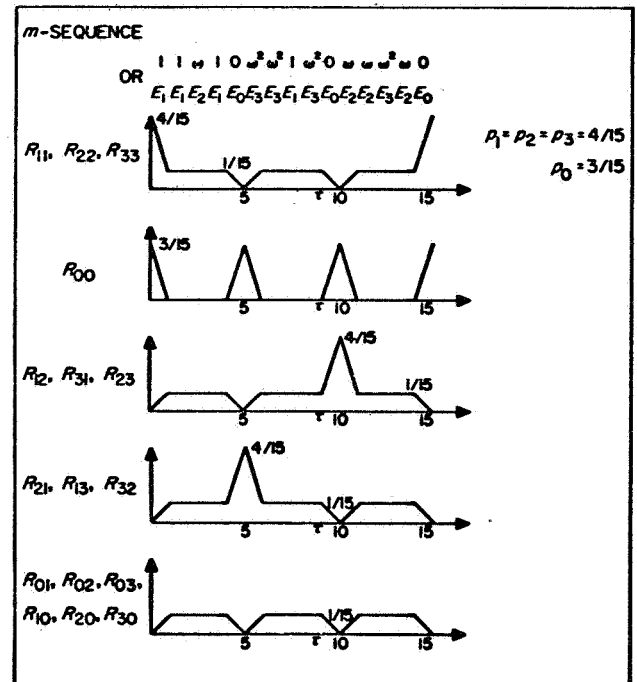


Fig. 7. Period-Normalized Correlation Functions for the Linear Sequence $b_n = b_{n-1} + \omega^2 b_{n-2}$ Over One Period

for all non-zero elements b_i of the sequence. The weighting functions $G_{ik}(f)$ are

$$G_{ik}(f) = \frac{1}{l_0} \left[p_i p_{ik} \left(\sum_{n=-\infty}^{+\infty} e^{-j\omega n l_0} - e^{-j\omega n l_0} \right) + p_i \sum_{n=-\infty}^{+\infty} \delta(k, \lambda^n i) e^{-j\omega n l_0} \right] \quad (67)$$

Because of the Kronecker delta in this Equation, the last term is actually

$$\sum_{r=1}^{a-1} p_i e^{-j\omega r l_0} \delta(k, \lambda^r i) \sum_{n=-\infty}^{+\infty} e^{-j\omega n l_0} \quad (68)$$

Notice that when $v = a - 1$, $\delta(k, \lambda^{a-1} i) = \delta_{ik}$. If either i or k is the zero, then $\delta(k, \lambda^v i) = \delta_{ik}$ for all v . The spectral density of the modulated m -sequence is

$$\begin{aligned} G(f) &= \frac{1}{l_0^2} \left\{ \frac{1}{p} \left[\left(\frac{p+1}{a^2} \right) \sum_{i=1}^a \sum_{k=1}^a H_i^*(f) H_k(f) - |H_a(f)|^2 \right] \right. \\ &\quad \times \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{l_0}\right) \\ &\quad + \frac{p+1}{p^2} \left(\frac{a-1}{a} \right) \left[|H_a(f)|^2 - \frac{1}{a} \sum_{i=1}^a \sum_{k=1}^a H_i^*(f) H_k(f) \right] \\ &\quad \times \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{l_0}\right) \\ &\quad \left. + \frac{p+1}{p^2} \left[\sum_{i=1}^{a-1} \sum_{k=1}^{a-1} \sum_{r=1}^{a-1} H_i^*(f) H_k(f) e^{-j\omega r l_0} \delta(k, \lambda^r i) \right] \right. \\ &\quad \times \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{l_0}\right) \left. \right\} \quad (69) \end{aligned}$$

When the sampled set $\{h_i(t)\}$ contains both $h_i(t)$ and $-h_i(t)$ for all $i \leq a$, then the sums $\sum_{i=1}^a \sum_{k=1}^a H_i^*(f) H_k(f)$ vanish; this simplifies the spectrum somewhat, but in general, cross terms still remain in the triple-sum term. When such a simplification is made, the process is said to be NEF (negative equally frequent).

As a special case, let $a = 2$ (PN sequence) with $H_1(f) = -H_2(f)$. Then

$$G(f) = \frac{p+1}{p^2 l_0^2} |H_1(f)|^2 \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{l_0}\right) - \frac{p}{p^2 l_0^2} |H_1(f)|^2 \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{l_0}\right) \quad (70)$$

If the PN sequence has a long period, the spectrum appears to be the Markov process of two equally likely states.

Figure 8 compares the binary NEP spectrum to that experimentally obtained from a PN sequence.

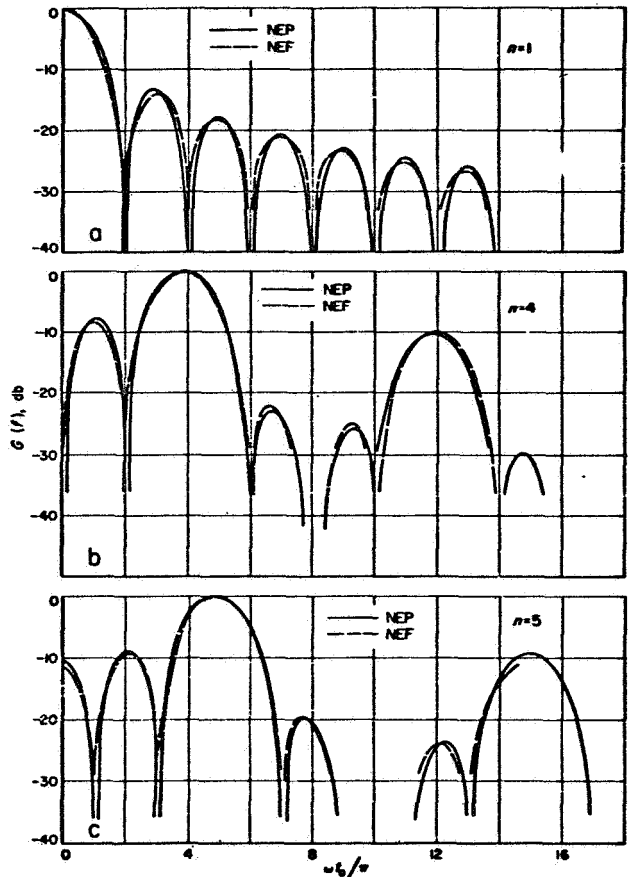


Fig. 8. Comparison of Normalized NEP and NEF Spectra for the Case $a = 2$

C. The NEFS Process

The preceding Part has shown that, under certain conditions, the Markov NEP process was approximated to an amazing degree. The spectrum of the periodic coun-

terpart of the NEPS process is determined by a somewhat cruder technique; the envelope is slightly in error, but the error decreases as the period becomes long. This less elegant approximation is adopted in order that a linearly recurring m -sequence may be used as a random-like sampler of $\{h_i(t)\}$. Although it may be possible to generate a sequence which will give results which look more like a Markov process, m -sequences have been previously discussed and will, therefore, be used in this treatment. A periodic process is said to be NEFS (negative equally frequent-same sign slope) if (1) both $h_i(t)$ and $-h_i(t)$ are in $\{h_i(t)\}$ and their frequencies of occurrence are nearly equal; (2) the transition frequencies between any two states are approximately equal to the transition frequencies between the two corresponding negative states; and (3) the slope does not change sign at transition times.

Sequences are restricted to those in which errors in frequencies of transition and occurrence stated above in (1) and (2) decrease as the period lengthens.

In choosing m -sequences to modulate $\{h_i(t)\}$, each of the a_i states E_i of the m -sequence chooses between $h_i(t)$ and $-h_i(t)$, selecting the one which has the proper slope for transition. In this way, the set $\{h_i(t)\}$ must have $a = 2a_1$ elements, q of which are even waveforms and r of which are odd. Both q and r are even-integers; also, a_1 is chosen to be even, and $a_1 = 2^k$ for some positive integer k . The period of the process depends on whether an even or odd number of odd waveforms occur per m -sequence period. If the number of odd waveform occurrences is even, the NEFS period equals the m -sequence period; otherwise, it is twice as great. The only state which occurs an odd number of times in the m -sequence is E_a ; therefore, $h_a(t)$ is set to be an even waveform, in order that the NEFS period is p , the m -sequence period.

Because of the postulated symmetries (2), the analysis given to NEPS processes also applies here, and Eq. (48) describes the spectrum involved, with the exception that the matrix $E(z)$ is

$$E(z) = \sum_{n=1}^m E^{(n)} z^n$$

$$E^{(n)} = B^{(n)} - C^{(n)} \quad (71)$$

These matrices $E^{(n)}$ are, of course, periodic, with elements $p_{ik}^{(n)} = p_{i,k+a/2}^{(n)}$. When $n \equiv 0 \pmod p$, $E^{(n)}$ is the identity matrix. When $n \equiv 0 \pmod s = p/(a-1)$, the matrix elements are 1, 0, or -1, depending upon i , k , and n . Using previous notation,

$$p_{ik}^{(n)} - p_{i,k+a/2}^{(n)} = \delta(k, \lambda^n i) - \delta\left[k, \lambda^n \left(i - \frac{a}{2}\right)\right] \quad (72)$$

For other values of n , $E^{(n)}$ behaves as

$$E^{(n)} = \left(\frac{q-r}{q+r}\right)^{n-1} E + \epsilon^{(n)}; \quad 1 \leq n < s \quad (73)$$

Elements of the error matrix $\epsilon^{(n)}$ have magnitudes on the order of $1/p$ for large p and $n \leq m$. (See Appendix D for discussion.) This is the type of expression obtained for $E^{(n)}$ of the aperiodic process, except for some small error. When at least one even and one odd waveform pair are present in the set $\{h_i(t)\}$, and E_a represents an even pair, then the values for $p_{ik}^{(n)} - p_{i,k+a/2}^{(n)} \approx a/p$ due to the fact that for long periods, roughly half the waves are reversed per period, and at intervals of (vs) the m -sequence correlations $R_{ik}(vs)$ are maximum or else totally uncorrelated (Fig. 7). For long periods, these terms are neglected, and the power distribution is

$$G(f) = \frac{2(p+1)}{ap^2f_0} \left\{ \left[\sum_{i=1}^a p_i |H_i(f)|^2 \right] \left[\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{pf_0}\right) \right] \right.$$

$$+ \frac{a-2}{a} \operatorname{Re} \left[\frac{e^{-j\omega t_0}}{1 - \left(\frac{q-r}{q+r}\right) e^{-j\omega t_0}} \right]$$

$$\times \left[\sum_{\alpha} H_{\alpha}^*(f) - \sum_{\beta} H_{\beta}^*(f) \right]$$

$$\times \left[\sum_{\alpha} H_{\alpha}(f) + \sum_{\beta} H_{\beta}(f) \right]$$

$$\times \left. \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{sf_0}\right) \right\} \quad (74)$$

where the sums over α and β include positive waveforms only.

The assumptions which have been made are that the period is long, so that all p_i are equal (within $1/p$), all non-zero $p_{ik}^{(n)}$ are equal (within $1/p$), and that $[(q-r)/(q+r)]^m \leq 1$.

As a special case, let $q = r$, with a binary PN sequence operating in the fashion described. Then $s = p$ and $a = 4$. Equation (74) simplifies to the approximate spectral density of the process,

$$G(f) = \frac{p+1}{2p^2f_0^2} \left\{ \sum_{i=1}^p |H_i(f)|^2 + \frac{1}{2} \operatorname{Re} \left[|H_e(f)|^2 - |H_o(f)|^2 + [(H_e^*(f) H_o(f) - H_o^*(f) H_e(f)) e^{-j\omega t_0}] \right] \times \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{st_0}\right) \right\} \quad (75)$$

Here again, the envelope of the spectrum is seen to be governed by a Markov process, within a constant multiplier, and the spectra appear to be the same to receivers whose bandwidths are greater than $1/pt_0$.

Because of this "envelope" relation between the two processes, the spectra of Examples 3 and 4 are also examples of the NEFS process, when weighted by the proper constant and multiplied by the delta-function series.

Figure 9 compares the $a = 4$ NEPS spectrum to that experimentally obtained by PN modulation.

D. Other Random-like Sequences

Comparison of the delta-function envelopes of the binary NEF and NEFS processes to the spectra of corresponding random Markov processes shows that they are the same, within a constant multiplier; also, in the limit, as the periods become very great, the two become the same, indicating that the long PN sequence is, as far as its spectrum is concerned, very nearly random.

The PN sequence, by definition, has been set to be a binary m -sequence and, as such, can only choose between two signals in a modulating set. Since the PN sequence is easy to generate, selection of waveforms in $\{h_i(t)\}$ by means of Boolean functions of the shift-register states in

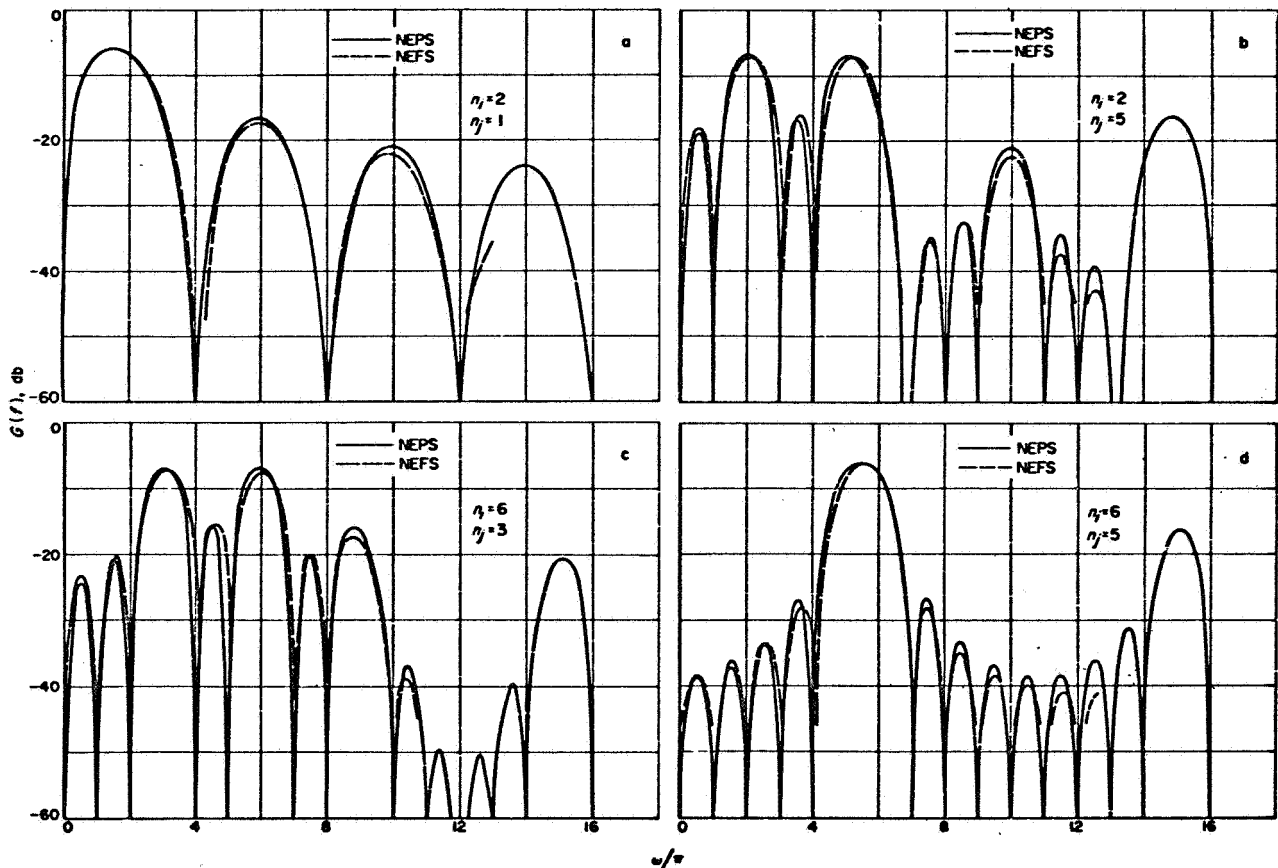


Fig. 9. Comparison of NEPS and NEFS Spectra for the Case $a = 4$

a PN sequence (word detection) is sometimes desirable. However, because of the cycle-and-add property (Eq. 63), these states may not have good correlation properties, and the spectra obtained in this sampling may be totally unlike their Markov counterparts (as is also true of the general m -sequence, $a > 2$). A better solution to this problem is to provide word detection on several independent PN generators of different periods.

In most cases, these PN sequences, or arrays of PN sequences, produce spectra comparable in some fashion to Markov spectra. When such is the case, since detection by conventional means involves sampling of data over a bandwidth Δf , spectra obtained from both Markov processes and pseudo-Markov processes appear to be the same when the period of the pseudo-random sequence is large compared to the reciprocal receiver bandwidth $1/\Delta f$.

IV. EXPERIMENTAL RESULTS AND CONCLUSIONS

A. Experimental Results

The experiments based upon the preceding parts of this Report were performed basically to determine the theoretical appearance of the spectra and secondly, to verify by actual spectral measurement that the theoretical curves were achieved to some degree. The first experiments were performed on an electronic digital computer for the Markov NEP and NEPS sine and square-wave processes. Such experiments were carried out only on simple cases, in which the mathematics involved did not require long programming time or lengthy calculation within the computer itself. The results which were obtained are plotted as the Markov spectra of Figs. 5, 8, and 9.

All the processes investigated in this way were binary processes (i.e., the sequence was binary), and the carriers were either sinusoid or square waves. Only binary cases were considered for two reasons. First, for the NEP process, any a -level process is merely the weighted sum of binary processes (see Eq. 32); and second, the Markov and linear-sequence processes only correspond for binary sequences.

The linear m -sequence spectra were measured using a sweeping-oscillator type of spectrum analyzer. The actual pseudo-random modulated square-wave process was synthesized on a miniature laboratory digital computer, as shown in Fig. 10. This computer is basically a bank of dynamic "and" gates, "or" gates, and delay lines, readily programmed to form flip-flops, sequence generators (see Fig. 6), and carrier generators (square waves).

The period of the m -sequence was set so that the width between spectral lines of the process was less than the analyzer bandwidth. These analyzer curves are shown in Figs. 5 and 9 for both the NEF and NEFS processes, and are compared to their Markov NEP and NEPS counterparts.

These plots speak fairly well for themselves. They show that, as predicted mathematically, the random and pseudo-random spectra agree quite closely. It must be remembered that one of the curves (Markov) is calculated and that the other (m -sequence) is measured, and therefore some tolerance due to analyzer bandwidth is in order. Such a discrepancy was not thought to be great, and the error encountered was not even calculated.

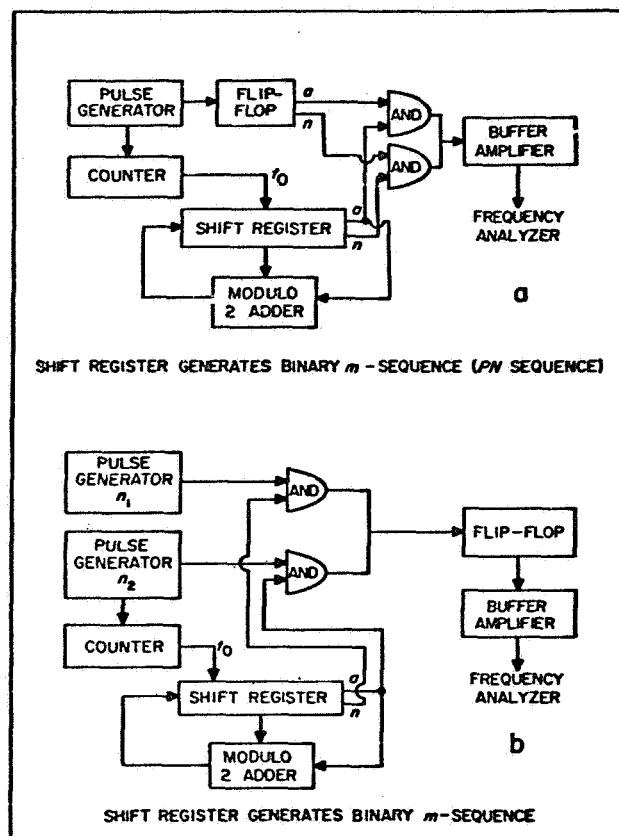


Fig. 10. Block Diagrams of NEF and NEFS Square-Wave Process Generators

B. Conclusions

Random and pseudo-random sequences may modulate signals in such a way that concentrations of power at specific frequencies are not apparent. If the autocorrelation functions of the two sequences are closely similar over a period of the pseudo-random sequence and the random sequence correlation is fairly constant outside this range, then the spectra resemble each other. Furthermore, if the period of the deterministic sequence is greater than the reciprocal of the receiver bandwidth, the periodic process has a spectrum which appears to be continuous and the same as that generated by a purely random process.

As simple as this seems, it may be shown that a periodic process never possesses exactly the proper correlation matrix for a Markovian spectrum. However, approxima-

tions to this behavior are possible, as has been shown for binary m -sequences. The correlations of m -sequences in general, however, do not look Markovian (see Fig. 7), and therefore, the spectrum of a process modulated by these sequences is not expected to look like a Markov spectrum. The problem encountered in attempting to make the two spectra appear the same is not an easy one. In the sense that general m -sequences do not offer the desired characteristics, Boolean functions of particular states in a binary m -sequence probably do not give desirable results either.

When transitions between waveforms are such that negative waveforms are as likely to occur as positive waveforms, the power spectrum of the corresponding process is merely the weighted sum of the energy spectra of the sampled waveforms. When these waveforms are

sinusoids, the phase shift between the sinusoid and sequence determines whether the spectrum decays at 6 or 12 db/octave, and the point at which a cross-over between 6 and 12 db/octave occurs. For square waves, the spectra decay at 6 db/octave.

When transitions are arranged so that slopes cannot change sign at transition, the spectra decay at the same rates, but larger, more pronounced peaks located near fundamental frequencies of $h_i(t)$ occur. This means that these processes are doubly important: first, from the point of view that such a process may be easier to mechanize by reason of the smoother transition, and second, from the point of view that it is possible to create a broad band spectrum which falls off rapidly outside the band. To a transmitter, this means that power is not wasted outside the desired band.

NOMENCLATURE

- a = number of states.
 A = column vector of the positive waveforms in the frequency domain.
 B, C, D, E = matrices used to derive the NEPS spectrum.
 b_i = sequence element chosen from finite field K .
 c_i = feedback coefficients for the linear sequence.
 E_i = i th state of the sequence.
 $E[X]$ = mathematical expected value of X .
 f = frequency.
 $G(f)$ = output spectral density.
 $G_{ik}(f)$ = cross-spectral density of the sequence states.
 $h_i(t)$ = member of a set of modulated signals.
 $H_i(f)$ = Fourier transform of $h_i(t)$.
 H = column vector of the frequency-domain states.
 i, j, k, m, r, s, v = integer-valued indices.
 I_k = $k \times k$ identity matrix.
 j = $\sqrt{-1}$.
 K = finite field.
 L = length of the sequence.
 n = number of half-cycles per switching period.
 $N_{ij}^{(n)}$ = number of times state E_j occurs at n th transition after E_i .
 p = period of the sequence.
 $p_{ij}^{(n)}$ = relative frequencies at which state E_j occurs at n th transition after E_i .
 $P_{ik}(z)$ = generating function.
 $P^{(n)}$ = n th-transition matrix.
 $P(z)$ = generating matrix.
 $R_{ik}(\tau)$ = cross-correlation function.
 $R(m)$ = correlation matrix.
 s = $p/(a - 1)$.
 S = sequence.
 t = time.
 t_0 = switching period.
 $x(t)$ = a periodic process.
 $y(t)$ = output signal.

NOMENCLATURE (Cont'd)

$Y_T(f)$ = Fourier transform of $y_T(t)$.

$\alpha, \beta, \gamma, \epsilon$ = subscripts of the waveforms $h_i(t)$ denoting class.

$\delta(f)$ = Dirac delta-function.

Δ = increment.

$\Delta_i(f)$ = Fourier transform of $\delta_i(t)$.

ϵ = an arbitrarily small quantity.

ζ = convolution variable.

λ = ratio of spike to continuous spectral power.

λ = primitive element of field K .

τ = correlation variable.

ϕ = phase angle.

$\omega = 2\pi f$ = angular frequency.

$*$ = complex conjugation.

APPENDIX A

Derivation of the Markov Spectrum

Proof of Eq. (28): The signal $y(t)$ is obviously a measurable function of t . Under this condition, if

$$\phi(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} y(t) y(t + \tau) dt \quad (\text{A-1})$$

exists for all τ , then there exists an integrated power spectrum¹

$$S(\omega) = \frac{2}{\pi} \int_0^{\infty} \phi(\tau) \frac{\sin(\omega\tau)}{\tau} d\tau; \quad \omega = 2\pi f \quad (\text{A-2})$$

If $y(t)$ were a continuous stationary process, it is well known that $\phi(\tau) \equiv R(t, \tau) = E[y(t)y(t + \tau)]$ with probability 1. However, $R(t, \tau)$ in the present case is not independent of t . In Appendix B, it is proved that $\phi(\tau) \equiv R(\tau)$ with probability 1, where

$$\begin{aligned} R(\tau) &= R(nt_0 + t) \\ &= \frac{1}{t_0} \sum_{i,k=1}^{\infty} a_{ik}(n) \int_{-\infty}^{\infty} b_i(t_1) b_k(t_1 + t) dt_1 \\ &\quad + a_{ik}(n+1) \int_{-\infty}^{\infty} b_i(t_1) b_k(t_1 + t - t_0) dt_1 \end{aligned} \quad (\text{A-3})$$

and

$$a_{ik}(n) = \begin{cases} p_i \delta_{ik} & \text{if } n = 0 \\ p_i p_{ik} & \text{if } n > 0 \\ p_k p_{ki}^{(n)} & \text{if } n < 0 \end{cases} \quad (\text{A-4})$$

Here, τ has been decomposed into an integer times t_0 plus a part t with $0 \leq t < t_0$. The coefficients $p_{ik}^{(n)}$ consist of periodic terms plus terms which tend to zero exponentially (Ref. 4). Correspondingly, $R(\tau)$ will have periodic components plus a part which has a finite integral of its absolute value. The first part corresponds to jumps in $S(\omega)$, and the second part corresponds to a part of $S(\omega)$ which is the integral of a spectral density (Theorem of Plancherel).

¹See Ref. 2 (Chapter II, Section 3). In the present case, it can be shown that $y(t)$ also possesses an integrated spectrum (Chapter III, Sections 4 and 5).

The Periodic Part: For aperiodic Markov chains (see Ref. 4, Chapter XV), $p_{ik}^{(n)} \rightarrow p_k$, so that the periodic part is

$$\begin{aligned} &\sum_{i,k=1}^{\infty} \frac{p_i p_k}{t_0} \left[\int_{-\infty}^{\infty} b_i(t_1) b_k(t + t_1) dt_1 \right. \\ &\quad \left. + \int_{-\infty}^{\infty} b_i(t_1) b_k(t + t_1 - t_0) dt_1 \right] \end{aligned} \quad (\text{A-5})$$

for $0 \leq t \leq t_0$, and the transform is

$$\begin{aligned} a_n &= \sum_{i,k=1}^{\infty} \frac{p_i p_k}{t_0} \left[\frac{1}{t_0} \int_0^{t_0} \int_{-\infty}^{\infty} b_i(t_1) b_k(t + t_1) e^{-\frac{j2\pi n t}{t_0}} dt_1 dt \right. \\ &\quad \left. + \frac{1}{t_0} \int_0^{t_0} \int_{-\infty}^{\infty} b_i(t_1) b_k(t + t_1 - t_0) e^{-\frac{j2\pi n t}{t_0}} dt_1 dt \right] \end{aligned} \quad (\text{A-6})$$

In the second integral, if we replace t by $t + t_0$,

$$a_n = \sum_{i,k=1}^{\infty} \frac{p_i p_k}{t_0^2} \int_0^{t_0} \left[\int_{-\infty}^{\infty} b_i(t_1) b_k(t + t_1) e^{-\frac{j2\pi n t}{t_0}} dt_1 \right] dt \quad (\text{A-7})$$

The expression in brackets is zero for $|t| > t_0$, so that the limits can be replaced by $\pm \infty$, and it immediately follows that

$$a_n = \sum_{i,k=1}^{\infty} \frac{p_i p_k}{t_0^2} H_i^* \left(\frac{2\pi n}{t_0} \right) H_k \left(\frac{2\pi n}{t_0} \right) = \frac{1}{t_0^2} \left| \sum_{i=1}^{\infty} p_i H_i \left(\frac{2\pi n}{t_0} \right) \right|^2 \quad (\text{A-8})$$

The Spectral-Density Part: The corresponding part of $R(nt_0 + t)$ is

$$\begin{aligned} &\frac{1}{t_0} \sum_{i,k=1}^{\infty} [a_{ik}(n) - p_i p_k] \int_{-\infty}^{\infty} b_i(t_1) b_k(t + t_1) dt_1 \\ &\quad + [a_{ik}(n+1) - p_i p_k] \int_{-\infty}^{\infty} b_i(t_1) b_k(t_1 + t - t_0) dt_1 \end{aligned} \quad (\text{A-9})$$

The integral which gives the Fourier transform may be split into intervals to give

$$G(\omega) = \frac{1}{t_0} \sum_{i,k=1}^a \sum_{n=-\infty}^{+\infty} \left\{ [a_{ik}(n) - p_i p_k] e^{-j\omega n t_0} \right. \\ \times \int_{-\infty}^{+\infty} \int_n^{t_0} b_i(t_1) b_k(t + t_1) e^{-j\omega t} dt dt_1 \\ \left. + [a_{ik}(n+1) - p_i p_k] e^{-j\omega (n+1)t_0} \right. \\ \times \left. \int_{-\infty}^{+\infty} \int_0^{t_0} b_i(t_1) b_k(t + t_1 - t_0) e^{-j\omega t} dt dt_1 \right\} \quad (\text{A-10})$$

In the second integral, replacing n by $n+1$ and t by $t+t_0$, and using the same argument used in evaluating a_n , gives

$$G(\omega) = \frac{1}{t_0} \sum_{i,k=1}^a \sum_{n=-\infty}^{+\infty} [a_{ik}(n) - p_i p_k] e^{-j\omega n t_0} H_i^*(\omega) H_k(\omega) \quad (\text{A-11})$$

Letting $e^{-j\omega n t_0} = z^n$, the infinite series becomes (see Ref. 4, Chapter 16)

$$p_i (\delta_{ik} - p_k) + p_i \left[P_{ik}(z) - \frac{p_k z}{1-z} \right] \\ + p_k \left[P_{ki}(z^{-1}) - \frac{p_i z^{-1}}{1-z^{-1}} \right] \\ = p_i \delta_{ik} + p_i P_{ik}(z) + p_k P_{ki}(z^{-1}) \quad (\text{A-12})$$

Finally,

$$G(\omega) = \frac{1}{t_0} \sum_{i,k=1}^a [p_i \delta_{ik} + p_i P_{ik}(z) + p_k P_{ki}(z^{-1})] H_i^*(\omega) H_k(\omega) \quad (\text{A-13})$$

The theorem follows from this and the expression for the a_n .

APPENDIX B

Derivation of the Formula for Time Autocorrelation

Define

$$\phi(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} y(t) y(t+\tau) dt \quad (\text{B-1})$$

Since $y(t)$ is bounded, the limit exists if, and only if, the following limit exists:

$$\lim_{N \rightarrow \infty} \frac{1}{2Nt_0} \int_{-Nt_0}^{+Nt_0} y(t) y(t+\tau) dt \quad (\text{B-2})$$

If τ is written in the form $\tau = nt_0 + t_1$, where n is an integer, such that $0 \leq \tau - nt_0 < t_0$, and the integral is decomposed into intervals of length t_0 , then

$$\phi(nt_0 + t_1) = \lim_{N \rightarrow \infty} \frac{1}{2Nt_0} \sum_{m=-N}^{N-1} \int_0^{t_0} y(mt_0 + t) y[(m+n)t_0 + t + t_1] dt \quad (\text{B-3})$$

Now define

$$D_i(m) = \begin{cases} 1 & \text{if the state during } [mt_0, (m+1)t_0] \text{ is } E_i \\ 0 & \text{otherwise} \end{cases}$$

then

$$y(mt_0 + t) = \sum_{i=1}^a D_i(m) b_i(t) \quad (\text{B-4})$$

and

$$\phi(nt_0 + t_1) = \lim_{N \rightarrow \infty} \frac{1}{2Nt_0} \sum_{i,k=1}^a \sum_{m=-N}^{N-1} D_i(m) D_k(m+n) \\ \times \int_0^{t_0-t_1} b_i(t) b_k(t+t_1) dt + D_i(m) D_k(m+n+1) \\ \times \int_{t_0-t_1}^{t_0} b_i(t) b_k(t+t_1-t_0) dt \quad (\text{B-5})$$

There are a finite number of terms of the form

$$\sum_{m=-N}^{N-1} \frac{D_i(m) D_k(m+n)}{2N}$$

and their coefficients are constants as far as the variables m , n , and N are concerned, so that the limit becomes

$$\begin{aligned} \phi(mt_0 + t_1) &= \frac{1}{t_0} \sum_{i,k=1}^a \left[\lim_{N \rightarrow \infty} \sum_{m=-N}^{N-1} \frac{D_i(m) D_k(m+n)}{2N} \right] \\ &\times \int_{-\infty}^{+\infty} b_i(t) b_k(t+t_1) dt \\ &+ \frac{1}{t_0} \sum_{i,k=1}^a \left[\lim_{N \rightarrow \infty} \sum_{m=-N}^{N-1} \frac{D_i(m) D_k(m+n+1)}{2N} \right] \\ &\times \int_{-\infty}^{+\infty} b_i(t) b_k(t+t_1-t_0) dt \end{aligned} \quad (B-6)$$

The limits on the integrals can be made infinite because the integrands vanish outside the actual range of integration.

Now, suppose $n = 0$; then $D_i(m) D_k(m+0) = \delta_{ik} D_i(m)$

$$\lim_{N \rightarrow \infty} \sum_{m=-N}^{N-1} \frac{\delta_{ik} D_i(m)}{2N} = p_i \delta_{ik} \quad (B-7)$$

with probability 1, because the strong law of large numbers holds (Ref. 4). If $n > 0$, the process with state descriptions

$$E_{i_0, i_1, \dots, i_n} = (E_{i_0}, E_{i_1}, \dots, E_{i_n}) \quad (B-8)$$

is a Markov process with stationary distributions

$$Pr[E_{i_0, i_1, \dots, i_n}] = p_{i_0} p_{i_0 i_1} p_{i_1 i_2} \dots p_{i_{n-1} i_n} \quad (B-9)$$

The probability that $D_i(m) D_k(m+n)$ is one is

$$\sum_{i_1, \dots, i_n} Pr[E_{i_1}, E_{i_2}, \dots, E_{i_n}] \quad (B-10)$$

But that is just p_i times p_i^n . Using the strong law of large numbers, this Markov chain yields

$$\lim_{N \rightarrow \infty} \sum_{m=-N}^{N-1} \frac{D_i(m) D_k(m+n)}{2N} = p_i p_{ik}^n \quad (B-11)$$

with probability 1. A similar discussion applied to the case $n < 0$ gives

$$\lim_{N \rightarrow \infty} \sum_{m=-N}^{N-1} \frac{D_i(m) D_k(m+n)}{2N} = p_k p_{ki}^{-n} \quad (B-12)$$

with probability 1.

Define

$$a_{ik}(n) = \begin{cases} p_i \delta_{ik} & \text{if } n = 0 \\ p_i p_{ik}^n & \text{if } n > 0 \\ p_k p_{ki}^{-n} & \text{if } n < 0 \end{cases} \quad (B-13)$$

Then for each n ,

$$Pr \left[\lim_{N \rightarrow \infty} \sum_{m=-N}^{N-1} \frac{D_i(m) D_k(m+n)}{2N} = a_{ik}(n) \right] = 1 \quad (B-14)$$

But there is only a countable set of values of n , so that

$$Pr \left[\text{for all } n, \lim_{N \rightarrow \infty} \frac{1}{2N} \sum_{m=-N}^{N-1} \frac{D_i(m) D_k(m+n)}{2N} = a_{ik}(n) \right] = 1 \quad (B-15)$$

Defining

$$\begin{aligned} R(mt_0 + t_1) &= \frac{1}{t_0} \sum_{i,k=1}^a a_{ik}(n) \int_{-\infty}^{+\infty} b_i(t) b_k(t+t_1) dt \\ &+ a_{ik}(n) \int_{-\infty}^{+\infty} b_i(t) b_k(t+t_1-t_0) dt \end{aligned} \quad (B-16)$$

the preceding statement becomes $Pr[R(\tau) \equiv \phi(\tau)] = 1$.

APPENDIX C

Derivation of the NEPS Spectrum

Assume that the modulating subsets $\{h_a\}$ and $\{h_\beta\}$ contain q even states and r odd states, and designate H to be the column vector of the frequency-domain states:

$$H = \begin{bmatrix} H_a \\ H_\beta \\ -H_a \\ -H_\beta \end{bmatrix} = \begin{bmatrix} I_k \\ -I_k \end{bmatrix} A \quad (C-1)$$

where I_k is the unit matrix of dimension $k = a/2$, and A is the column vector

$$A = \begin{bmatrix} H_a \\ H_\beta \end{bmatrix} \quad (C-2)$$

Since the stationary probability for $h_i(t)$ is the same as for $-h_i(t)$, designate the vector pA to be

$$pA = \begin{bmatrix} p_a H_a \\ p_\beta H_\beta \end{bmatrix} \quad (C-3)$$

The quantity to be evaluated is $(pH^*)^T P(z) H$, which appears in the third term of Eq. (31). Clearly,

$$(pH^*)^T P(z) H = (pA^*)^T (I_k, -I_k) P(z) \begin{bmatrix} I_k \\ -I_k \end{bmatrix} A \quad (C-4)$$

The matrix $P(z)$ is evaluated from P by Eq. (25). Since the P matrix is of the form

$$P = \begin{bmatrix} B & C \\ C & B \end{bmatrix} \quad (C-5)$$

it may be verified that P^* is also of the same symmetric form. Therefore the generating $P(z)$ matrix is symmetric also.

$$P(z) = \begin{bmatrix} B(z) & C(z) \\ C(z) & B(z) \end{bmatrix} \quad (C-6)$$

Substitution of Eq. (C-6) into Eq. (C-4) gives

$$(pH^*)^T P(z) H = 2(pA^*)^T [B(z) - C(z)] A \quad (C-7)$$

which is now an equation involving matrices of order $k = a/2$ instead of a -th-order ones. Define matrices D and E to be

$$D = B + C \quad E = B - C \quad (C-8)$$

so that

$$P = \frac{1}{2} \left\{ D \begin{bmatrix} I_k & I_k \\ I_k & I_k \end{bmatrix} + E \begin{bmatrix} I_k & -I_k \\ -I_k & I_k \end{bmatrix} \right\} \quad (C-9)$$

By direct calculation,

$$P^* = \frac{1}{2} \left\{ D^* \begin{bmatrix} I_k & I_k \\ I_k & I_k \end{bmatrix} + E^* \begin{bmatrix} I_k & -I_k \\ -I_k & I_k \end{bmatrix} \right\} \quad (C-10)$$

Also

$$P(z) = \frac{1}{2} \left\{ D(z) \begin{bmatrix} I_k & I_k \\ I_k & I_k \end{bmatrix} + E(z) \begin{bmatrix} I_k & -I_k \\ -I_k & I_k \end{bmatrix} \right\} \quad (C-11)$$

where it is noted that

$$B(z) - C(z) = E(z) \quad (C-12)$$

and hence,

$$(pH^*)^T P(z) H = 2(pA^*)^T E(z) A \quad (C-13)$$

But from Eq. (C-8), it is known that

$$E = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\epsilon \end{bmatrix} \quad (C-14)$$

and, therefore, $E(z) = \sum_{n=1}^{\infty} E^n z^n$; then,

$$(pH^*)^T P(z) H = 2[p_a H_a^*, p_\beta H_\beta^*] E(z) \begin{bmatrix} H_a \\ H_\beta \end{bmatrix} \quad (C-15)$$

Substitution of this into Eq. (31) gives the desired result (Eq. 48).

APPENDIX D

Derivation of the NEFS Spectrum

Consider the process in which an a_1 level m -sequence samples the set $\{h_i(t)\}$ in such a fashion that each term of the sequence E_i selects either $h_i(t)$ or $-h_i(t)$, so that the slope does not change sign in transition. Each of the a_1 states occurs a_1^{m-1} times, except state E_{a_1} itself, which only occurs $a_1^{m-1} - 1$ times. These differ by only one part per period. Similarly, the states of the NEFS occur $\frac{1}{2}(a_1^{m-1})$ and $(\frac{1}{2}a_1^{m-1} - 1)$ times per period.

In the m -sequence, the states $E_i E_j E_m E_k$ occur in sequence a_1^{m-4} times (if all $i, j, m, k \neq a_1$). Select E_i and E_k to be a specific transition, and assume that E_i is a positive even waveform and E_k is a positive waveform. Then such a transition occurs

$$\begin{aligned} N_{ik}^{(1)} &= \frac{1}{16} a_1^{m-4} (q^2 + r^2) \\ N_{i,k+a_1}^{(3)} &= \frac{1}{16} a_1^{m-4} (2qr) \end{aligned} \quad (D-1)$$

per period; hence,

$$\begin{aligned} E_{ik}^{(1)} &= p_{i,k+a_1}^{(3)} - p_{i,k}^{(3)} = \frac{a_1^{m-4} (q-r)^2}{16 (a_1^m - 1)} \\ &= \left(\frac{q-r}{q+r} \right)^2 (p_{ik} - p_{i,k+a_1}) \end{aligned} \quad (D-2)$$

By symmetry, this is also true for all $E_{ik}^{(1)}$. This error in making the above statements is on the order of $1/p$. Similar reasoning shows that this approximation is applicable up to $n = \pm m$ and beyond, within a few errors

also on the order of $1/p$. If these errors are neglected,

$$\begin{aligned} E(z) &= \frac{1}{I_0} \left[\sum_{k=1}^{a_1-1} E \left(\frac{q-r}{q+r} \right)^k z^k \right] \sum_{n=-\infty}^{+\infty} z^{na} \\ &\quad + \frac{1}{I_0} \left[\sum_{r=1}^{a_1-1} E^{(rs)} z^{(rs)} \right] \sum_{n=-\infty}^{+\infty} z^{na} \\ &\quad + \frac{1}{I_0} \left[\frac{1 - \left(\frac{q-r}{q+r} \right)^{a_1-1} z^{a_1-1}}{1 - \left(\frac{q-r}{q+r} \right) z} \right] E z \sum_{n=-\infty}^{+\infty} z^{na} \\ &\quad + \frac{1}{I_0} \left[\sum_{r=1}^{a_1-1} E^{(rs)} z^{(rs)} \right] \sum_{n=-\infty}^{+\infty} z^{na} \end{aligned} \quad (D-3)$$

Upon substitution of $z = e^{-j\omega t_0}$, the infinite series of exponentials becomes a series of delta-functions in the frequency domain. If the period is long $[(q-r)/(q+r)]^{a_1-1} \ll 1$, and therefore this term is negligible. Substituting this $E(e^{-j\omega t_0})$ into Eq. (48) gives the spectrum in Eq. (74).

It should be noted that errors encountered in this analysis are only estimated to be of the order $1/p$. For $E_{ik}^{(n)}$ up to $n = \pm m$, this estimation is fairly good. Beyond this point, certain sequences do not exist, and therefore decrease the factor multiplying E . This means that the factors $[(q-r)/(q+r)]^k$ are somewhat in error but are probably less than the estimated $[(q-r)/(q+r)]^k$. For this reason, it is felt that in dropping the term $[(q-r)/(q+r)]^{a_1-1}$, no great error is committed.

REFERENCES

1. Price, R., and Green, P. E., Jr., "Communications Techniques for Multipath Channels," *Proceedings of the IRE*, 46, No. 3:555-569, March 1958.
2. Wiener, N., "Generalized Harmonic Analysis and the Theory of Probability," *Acta Mathematica*, 55:117-258, 1930.
3. Davenport, W. B., Jr., and Root, W. L., *An Introduction to the Theory of Random Signals and Noise*, 1st Edition, McGraw-Hill Book Co., Inc., New York, 1958.
4. Feller, W., *Introduction to Probability Theory and its Applications*, Vol. I, John Wiley, Inc., New York, 1950.
5. Goldman, S. G., *Frequency Analysis, Modulation and Noise*, McGraw-Hill Book Co., Inc., New York, 1948.
6. Golomb, S. W., *Sequences with Randomness Properties*, Terminal Prog. Rep. under contract Req. No. 639498, Acct. No. 7570-505-739, Glen L. Martin Company, Baltimore, Maryland, June 14, 1955.
7. Zierler, N., *Linear Recurring Sequences*, Group Report 34-63 (Rev.), Massachusetts Institute of Technology, Lincoln Laboratories, Cambridge, Massachusetts, August 21, 1958.
8. Sittler, R. W., *Analysis and Design of Simple Nonlinear Noise Filters*, ScD Thesis, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1954.
9. Lorens, C. S., *Theory and Application of Flow Graphs*, ScD Thesis, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1956.

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